Mixing

THE TEACHING OF COLLEGE MATHEMATICS*

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1. Introduction. The subject I wish to discuss this afternoon is one with which we are all vitally concerned. Indeed now that I think about it, I realize that when I selected it I was guided by the demon of rashness rather than by the angel of caution. For in a roomful of teachers there are likely to be as many opinions as there are heads, and these opinions are not readily relinquished. Any proposals which I may make towards a change in our methods will appear to some of you tainted with the restless radicalism that is everywhere in the world around us; and the tolerant liberals amongst you will say with that kindly cynicism that is so terrible: The more you change it the more it remains ever the same. Nevertheless the matter seems to me of such fundamental importance for the welfare of our country that I shall proceed with my declared intention. If my remarks seem addressed to the young and beginning teachers among you, it is because I feel that in your hands lies the mathematical future of our students. To the older and more experienced teachers among you I simply remark that I too have the advantage of experience; for I am finishing this year my thirtieth year of teaching college mathematics.

The subject of my talk has two main aspects: What should we teach and how should we teach it? Both of these are important, but I am firmly convinced that the importance of the second is overwhelmingly greater than the importance of the first. I shall, then, say just a few words about what we should teach and shall devote myself mainly to the question of how our teaching should be done. You probably feel that the content of the Freshman course is largely determined by the high-school preparation of the students who come to college. In some colleges the first year is devoted to algebra and trigonometry, and in others to trigonometry and analytic geometry.

At Johns Hopkins, where I teach, we have two classes of entering students: those who enter with trigonometry and those who have not had trigonometry. Students of the first category devote the first sixteen-week term to analytic geometry and the second sixteen-week term to differential calculus, the classes meeting four hours weekly. Students who have not had trigonometry meet five hours weekly and cover trigonometry, analytic geometry, and differential calculus in the two sixteen-week terms of the Freshman year. The Sophomore year, consisting of two sixteen-week terms with classes meeting four hours weekly, is devoted to integral calculus and advanced parts of differential calculus and of analytic geometry. Beginning next fall a third year of mathematics is to be required for all our engineering students. In this third year the classes meet three hours weekly, and the subjects treated are differential equations, matrices and complex variable theory. For majors in mathematics, semester courses in modern algebra, differential equations, topology, and statistics are offered.

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However, my interest today centers on the first two years, and I shall not discuss the more advanced courses.

2. The Freshman course. What should be the content of the Freshman course? In this friendly family group I may be permitted to speak my mind frankly without thought or fear of giving offense. Where did this worship of algebra (I care not whether you call it college algebra or algebra unadorned) and of trigonometry arise? When some painstaking historian tries to assess the contribution of teachers of mathematics to the war effort he will have to say: They certainly made a gallant effort to teach algebra and trigonometry. The thousands of classes in ESMWT and ASTP courses which were devoted to algebra and trigonometry stagger the imagination. What did it all amount to? Forget the ordinary run-of-the-mine student who just managed to pass the course and think for a moment only of the two or three excellent students in each class who got 95 or 100. What does it profit them, from the point of view of mathematics, that they can manipulate identities, juggle half-angle formulas and solve triangles, even mastering the ambiguous case?

The one thing that is of mathematical value and interest in a trigonometry class is simply this: The cosine of an angle is a simpler concept than the angle itself. I have to confess that I have never been fortunate enough to meet a student who has passed a trigonometry class and who has grasped this one essential fact. When I ask such students what a right angle is they reply without exception that it is 90 degrees; and when I ask what a degree is they are always pleased to be able to tell me that it is one-ninetieth part of a right angle. There is something malevolently wrong and hypnotic about a course which permits a bright, alert young American who can detect sham in many matters to hoodwink himself in this way.

My first suggestion, then, is that you forget your complaints about the poor preparation in algebra and the lack of trigonometry in the high school. Start teaching analytic geometry in the first term of the Freshman year, but teach it by the methods of vector analysis. In this way you will not only be able to make the teaching of analytic geometry easier, but you will at the same time be teaching those concepts of algebra which are at once the simplest, the most fundamental and the most significant. Start with vectors on a line and your students will learn just what negative numbers are, and they will be pleased to find out just what is going on when they manipulate their slide rules. When you come to plane vectors you can teach, through the concept of direction cosines, the essential elements of trigonometry. For instance the central formula of trigonometry.

$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$

is nothing but a variant of the formula furnishing the distance between two points. When one remembers the complicated constructions involved in the usual proofs of this formula, one can only wonder at the genius for making simple things complicated. Before dealing with second degree curves in the plane,

such as the circle, ellipse, hyperbola, and parabola, take up the simple first degree problems in space. Introduce two-row and three-row determinants in a simple natural way as vector products.

When a student who has been taught analytic geometry by means of vectors looks at an equation such as $2x-3y+7=0$, his first reaction is that this is an equation of a line perpendicular to the coefficient vector $\mathbf{v}(2, -3)$. He knows that this coefficient vector serves to direct the line and that the positive side of the directed line is that towards which the coefficient vector points. The formula for the distance from a point (x, y) to the line $2x-3y+7=0$ gives him no trouble; all he has to do is to divide the expression $2x-3y+7$ by the magnitude $\sqrt{13}$ of the coefficient vector. Similarly when he looks at an equation such as $3x-2y+4z-11=0$ he knows at once that this is an equation of a plane perpendicular to the coefficient vector $\mathbf{v}(3, -2, 4)$ and that this coefficient vector serves to orient the plane (the positive side of the oriented plane being that towards which the coefficient vector points). He thus gets a clear understanding of the fact that it is without meaning to speak of the positive side of a line or of a plane; it is only when the line is directed, or the plane oriented, that it makes sense to speak of its positive side.

We have taught analytic geometry in this way at Hopkins for several years and we find that it goes over well. Our text has been published and so I shall merely refer you to this text for further details and shall pass on to the problem of teaching calculus. Before doing so, I throw out the suggestion that the elementary (or linear) part of analytic geometry, taught by the method of vectors, might well be given in the last year of high school. If, in order to do this, you have to jettison courses in trigonometry, algebra B or solid geometry, you may do this without loss. It is more important for a high school student, who may not go on to college, to know what a vector is, and how cosines facilitate the resolution of vectors, than it is for him to manipulate trigonometric identities, or to solve quadratic equations or to memorize the complicated constructions of solid geometry. A strong argument in favor of teaching the elementary parts of analytic geometry in high school is that the student who goes to college can begin his calculus in the second term of the Freshman year. The demands of modern physics, chemistry, and engineering for the elements of calculus are so pressing that it is no longer practicable to postpone the teaching of calculus to the Sophomore year.

3. The calculus. How should calculus be taught? Here I am very definitely in the minority, but I have the conviction of the true believer: I *know* that I am right. Most of my friends, in a kindly endeavor to prevent me from wasting my time, assure me that calculus simply cannot be taught correctly. The only thing to do is to teach manipulative technique; the student who can differentiate and integrate complicated expressions is at least not scared when he sees symbols for derivatives and integrals. The few students who can clearly understand what a derivative or an integral is will find this out later either in more advanced courses or by independent study. For many years I have at every opportunity

asked college teachers what text they use in their calculus courses and, as nearly as I can estimate, eighty percent of our teaching of calculus is done from a text which does not tell the student what a number is. Certainly ninety percent of the students who come to my graduate course in applied mathematics (for which courses in calculus and differential equations are prerequisite) have never heard of the upper and lower bounds of a bounded function and do not even seem to have heard the term bounded function. I have difficulty in finding words strong enough to express my disagreement with this opinion of my friends and my dissatisfaction with the resulting situation. The times are moving so fast that this situation simply cannot stand, and now is the moment to change it. Have courage, you young men of my audience who wish that things were different. They can be, and you will make them different. When I speak to a successful engineer he almost invariably says to me: For Heaven's sake, teach your young students understanding of the fundamental principles rather than manipulative detail. The latter they can gain for themselves, but they will never learn the former unless you teach it to them. It is my firm conviction, confirmed by several years of experience, that one can teach calculus correctly from the very beginning without losing the ordinary run-of-the-mine student and without sacrificing manipulative technique. We have been teaching calculus in this way at Hopkins for some four years from mimeographed notes, and we hope to have these published by next fall.* I trust that you will examine this text when it appears.

Why is it that so few students grasp the fundamental concepts of calculus in a first course? It is simply because they are deluged with terms such as limit, derivative, differential, integral, and they are never told what a number is. No student can hope to know what a limit is until he understands that every number is defined by a nested sequence of intervals. It takes energy on the part of the teacher to explain this and to make clear, for example, what $3^{1/2}-2^{1/2}$ means. But you are simply not worth your salary, and are certainly not earning it, if you have not the energy and the will to do this. It is quite unnecessary to use high-sounding phrases such as Dedekind cuts, but even poor students can be taught such a theorem as the one that says that a continuous function over a closed interval is bounded. They will make mistakes in repeating the proof, but this is no reason for lying awake at night worrying. They will have some idea of what calculus is about and will have at least a chance of understanding the Theorem of the Mean which is the central theorem of differential calculus. After all, calculus is the science of calculation, and the Theorem of the Mean is the theorem on which calculations are based. We may devoutly hope that students of calculus will in the future understand the paradox which is resolved by the Theorem of the Mean. All calculations involve a fateful leap from a safe initial point a to a final point b, and differentiation is a local process involving no such leaps. How, then, can calculations be made by means of derivatives?

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This is not the place to discuss in great detail isolated points in the teaching of calculus. I shall content myself with a few remarks which will serve to indicate how calculus should, in my opinion, be taught. In the first place I must protest against the universal use of the non-mathematical word small. Every present day student thinks that dx is a "small" increment of x. It is in fact unfair to a student to define the differential of a dependent variable or function and then to use, without explanation, a symbol for the differential of the independent variable. At this point the student who has learnt his analytic geometry by the method of vectors has a decided advantage. He knows that the tangent to the graph of a differentiable function $y=y(x)$ is simply the straight line through (x, y) which is parallel to the vector (dx, dy), where dx is arbitrary. This advantage is more marked when he passes to a study of functions of two or more variables. He sees at once that the tangent plane to the (surface) graph of the differentiable function $z=z(x, y)$ is simply the plane through (x, y, z) which is parallel to the vector $\mathbf{v}(dx, dy, dz)$, where dx and dy are arbitrary.

Most of the teaching of calculus is devoted to functions of a single variable, but most of the interesting applications to engineering, physics and chemistry have to do with functions of several variables. This makes it important that functions of a single variable should be treated in a manner that is readily extensible to functions of several variables. Instead of plunging at once into a discussion of the derivative, I find it advantageous to introduce the concept of a differentiable function; a differentiable function is one which may be linearized or straightened out in the neighborhood of a given point. A linear function $y=y(x)$ is one for which $\Delta y=c\Delta x$, where c is a constant, and a differentiable function is one for which $\Delta y=c\Delta x+\nu|\Delta x|$ where ν is a null-function of Δx, *i.e.*, a function whose limit at $\Delta x=0$ is zero. The constant function c of Δx is then simply the derivative of $y=y(x)$. Similarly a function $z=z(x, y)$ of two variables is differentiable if $\Delta z=c_1\Delta x+c_2\Delta y+\nu r$, where ν is a null-function of Δx and Δy and $r=\{(\Delta x)^2+(\Delta y)^2\}^{1/2}$. The differentials $dy=c\Delta x$ and $dz=c_1\Delta x+c_2\Delta y$ appear then in a natural way and the student understands that we always take differentials of functions or dependent variables (the differential dx of the independent variable x being merely a name for the differential of the identity function $y=x$ of x). This emphasis on Δy rather than on the derivative $y'=\lim \Delta y/\Delta x$ makes the proof of the important chain-rule theorem much simpler.

You all know that a critical point in the teaching of calculus occurs at the place where one proves that the limit at $x=0$ of $(\sin x)/x$ is 1; in other words, we prove that the derivative of $y=\sin x$ at $x=0$ is 1. This is a preliminary theorem to the proof of the fact that $y=\sin x$ is differentiable at any point x with derivative $\cos x$ and the transfer from $x=0$ to an arbitrary x involves a formula from trigonometry that most of your students have forgotten. The proof of the preliminary theorem proceeds as follows: It is clear that the length of a circular arc is greater than the chord and is less than the sum of the tangents at its end points. Once this is admitted, it is easy to prove the theorem. But are the two statements clear? if so, why not prove them? This proof cannot be furnished if

the term "length of an arc" is not defined. It seems to be the accepted opinion that this concept is too difficult for an intelligent young man of eighteen who is quite capable of piloting a B-29. Accepted or not, I hold the opinion absurd: the length of an arc is simply the upper bound of all chord sums. The only thing you have to prove is that the chord sums constitute a bounded variable; and since the length of each chord $\leqq |\Delta x| + |\Delta y|$ this is trivially evident for an arc for which y is a monotone function of x. It is more difficult to prove the additive property of arc length, but most students can follow the proof, even if they find difficulty in repeating it. Once the additive property has been proved, the proof of the relation $s_x^2 = 1 + y_x^2$ for a smooth arc by means of the Theorem of the Mean is immediate. Applying this to the circular arc we obtain at once the derivative of $y = \text{Arc cos } x$ and this gives (on using the theorem furnishing the derivative of the inverse of a given function whose derivative is known) the derivative of $y = \cos x$ and, hence of $y = \sin x$ at any point x. There is more to this than just a mere matter of taste. Once you permit yourself to shrink from a definition of the concept of arc length you have no secure foundation on which to build your treatment of such fundamental concepts as velocity, acceleration and curvature. The fact that some, at least, are disturbed by the claim that an arc of a circle is less than the sum of the tangents at its end-points is evidenced by the occasional use of inequalities between areas to obtain the bracketing inequalities which are necessary to prove that the limit at $x = 0$ of $(\sin x)/x$ is 1. You will agree with me, I believe, that it is a reversal of the direct order of procedure to use results on areas (area being naturally an undefined concept at this stage) to obtain results on arc lengths. It seems to me, then, indefensible to use shoddy arguments to derive a partial result (namely that $y = \sin x$ is differentiable at $x = 0$ with derivative 1) when the general result (namely that $y = \sin x$ is differentiable at any value of x with derivative $\cos x$) may be easily proved through the concept of the inverse function, particularly since this correct proof furnishes a solid basis for the applications of calculus to mechanics and engineering.

When you ask a student taught as I would have him taught the basic questions: What is a derivative? What is an integral? he will not reply: A derivative is the slope of a curve and an integral is the area under a curve. For he knows that the next questions will be: What is the slope of a curve and what is the area under a curve? No; he will define properly the basic terms derivative and integral and he will explain the terms slope and area by means of these.

The usual definition of an integral as $\lim_{\Delta x \to 0} \sum_{i=1}^{n} y_i \Delta_i x$ is open to serious objection. Remember that your defenseless student has been exposed only to limits of functions of a single variable; of what single variable is $\sum_{i=1}^{n} y_i \Delta_i x$ a function? Certainly not of Δx, which is merely a symbol for the greatest of the numbers $\Delta_i x$. The whole matter is simple if you have courageously grasped the nettle of upper and lower bounds. You simply introduce the familiar upper and lower sums $S = \sum_{i=1}^{n} M_i \Delta_i x$, $s = \sum_{i=1}^{n} m_i \Delta_i x$ and show that the variable S is bounded below, with lower bound B, and that the variable s is bounded above

with upper bound b. If $B=b$ the function y is integrable over the interval in question and the common value of B and b is its integral over this interval. The proof of the fact that any monotone function is integrable is so easy that even the poorest students learn to repeat the proof. One of the easiest ways to introduce the important function $y=e^x$ and its inverse function $y=\log x$ is to define the latter as the integral of the monotone function $1/x$ over the interval $[1, x]$, where $x>0$.

The first challenge I may expect from you is the following: In your effort to provide understanding, do you propose to sacrifice technique? My answer is no. The good teacher judiciously intermingles theory and practice. There is no doubt that some of the time at present devoted to elaborate differentiations and integrations must be given over to teaching correctly the underlying theory, but we find that even the poor students who find the theory difficult end up about as well trained in manipulation as those taught under the old system.

We stand at a critical juncture. Whether we like it or not, the torch of leadership in mathematical culture has been thrust into our hands. Hold it courageously aloft and do not let it drop to the ground to be picked up by some other nation that knows its value and the honor attached to it.

TWO THEOREMS ON FIBONACCI'S SEQUENCE

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Let u_n denote the nth member of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, \cdots \qquad (u_{n+1} = u_n + u_{n-1})$$

then the following theorem follows easily from known properties* of this sequence:

THEOREM 1. *Let the greatest common divisor (m, n) of m and n be 1, 2, or 5, then u_{mn} is divisible by $u_m u_n$.*

The object of the first part of this note is to show that the converse is also true:

THEOREM 2. *If u_{mn} is divisible by $u_m u_n$, then $(m, n)=1$, 2, or 5.*

The proof of Theorem 2 is based on the following well known results on Fibonacci's sequence.

Let $u_\alpha(\alpha>0)$ be the first member of the Fibonacci sequence divisible by a given prime p. Then $\alpha=\alpha(p)$ is called the rank of apparition of p and is some divisor of $p-(5/p)$.† Any term of the sequence u_r is divisible by p if and only if α divides r. Let p^λ be the highest power of p dividing u_α and let $r=p^\lambda k$, where p

* For proofs of the known results quoted in what follows see, for example, R. D. Carmichael, Annals of Math. (2), v. 15, 1913–14, pp. 30–70.

† The symbol $(5/p)$ is Legendre's symbol.

does not divide k, then Lucas' "law of repetition" for the Fibonacci sequence states that the highest power of p dividing u_r is $p^{\pi+\lambda+\gamma(r)}$ where

$$\gamma(r) = \begin{cases} 1 & \text{if } p^\lambda = 2 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that for a fixed p

$$\gamma(rs) \leqq \gamma(r) + \gamma(s). \tag{1}$$

To prove Theorem 2 suppose that d, the greatest common divisor of m and n is different from 1, 2 and 5. Now u_d is not a power of 5, since otherwise by the above paragraph with $p=5$, α would be 5, d would be divisible by 5 and would contain the same power of 5 as u_α, whereas $u_d > d$ for $d>5$. Therefore there exists a prime $p \neq 5$ dividing u_d since $u_d > 1$ for $d>2$. Then $\alpha(p)$, being a divisor of $p \pm 1$, is prime to p and is a factor of d. Let us write

$$d = \alpha h, \qquad m = m'd = m'\alpha h, \qquad n = n'd = n'\alpha h, \qquad mn = m'n'\alpha^2 h^2.$$

Finally let p^μ and p^ν be the highest power of p dividing m and n respectively, then the highest powers of p dividing u_m, u_n, u_{mn} and $u_m u_n$ are

$$p^{\mu+\pi+\gamma(m)}, \qquad p^{\nu+\pi+\gamma(n)}, \qquad p^{\mu+\nu+\pi+\gamma(mn)}, \qquad p^{\mu+\nu+2\pi+\gamma(m)+\gamma(n)}.$$

Since by assumption $u_m u_n$ divides u_{mn}we must have

$$\mu + \nu + \pi + \gamma(mn) \geqq \mu + \nu + 2\pi + \gamma(m) + \gamma(n),$$

or

$$\pi \leqq \gamma(mn) - \gamma(m) - \gamma(n).$$

But by (1) this is $\leqq 0$, which is absurd. Thus d must be 1, 2 or 5, which proves the theorem.

The second part of this note is devoted to the following theorem.

THEOREM 3. *Let $\alpha(p)$ be the rank of apparition of p. Then $[p-(5/p)]/\alpha(p)$ is an unbounded function of p.*

To prove this we need the following lemma.

LEMMA. *For every set $k_1, k_2, \cdots, k_s$ of integers >0 there exists a prime $q>5$ such that all the numbers*

$$k_1 q \pm 1,\ k_2 q \pm 1, \cdots, k_s q \pm 1$$

are composite.

Proof: By Dirichlet's theorem, the arithmetical progression

$$\{(3k_1 - 1)(3k_1 + 1) \cdots (3k_s - 1)(3k_s + 1)\}x + 3, \qquad x = 0, 1, 2, \cdots$$

contains at least one prime $q>5$ for let us say, $x = x_q$. But then

$$k_i q \pm 1 = \{(3k_1 - 1)(3k_1 + 1) \cdots (3k_s - 1)(3k_s + 1)x_q + 3\}k_i \pm 1$$

is divisible by $3k_i \pm 1$, a factor >1 and $<k_iq+1$, since $q \geqq 7$. Hence for all i, $k_i \pm 1$ are composite.

Proof of Theorem 3: Let q be a prime >5, and let p be a prime factor of u_q. Then $p = kq + (5/p) = kq \pm 1$. Since $\alpha(p) = q$, the function $[p-(5/p)]/\alpha(p) = k$. Suppose, if possible, that this function is bounded so that k can take on only the values

$$k_1, k_2, \cdots, k_s.$$

Then by the lemma the prime q can be chosen so that all $k_iq \pm 1$ are composite. This contradicts the fact that $kq \pm 1$ is a prime p, which proves the theorem.

DETERMINANTS WHOSE ELEMENTS ARE 0 AND 1

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1. Introduction. It was proved by Hadamard [1] that the maximum value of a determinant of order n, each element of which is not greater in absolute value than unity, is $n^{n/2}$. Further it was shown that this maximum value can be attained only, if each element of the determinant has the value ± 1 and if $n = 1$, 2, or $n \equiv 0 \bmod 4$. Here we investigate determinants of matrices K_n, each element of which has the value ± 1, when $n \not\equiv 0 \bmod 4$. In particular we show that, when $n = 7$, the maximum value for such a determinant is $2^9 9$ and that there is essentially only one type of determinant with this value. In the last two sections we construct special matrices K_n first, when $n = 4m$, and then, when $n = 9$, 10 and 11. Unfortunately it is impossible to tell by the methods used here whether or not these matrices K_n have determinants of maximum value. However, we do show that, when $n = 9$, 10 or 11, $|K_n|$ is greater in absolute value than any nth order minor of a determinant $|K_{12}|$ of maximum value 12^6.

2. Preliminary definitions. We shall say that two matrices K_n and K_n' are equivalent, $K_n \approx K_n'$, if K_n' can be obtained from K_n by a finite number of transformations of the following types: multiplication of any row by -1, interchange of any two rows, and transposition of the matrix. It follows that, if $K_n \approx K_n'$, the absolute values of $|K_n|$ and $|K_n'|$ are the same. Further there exists a matrix $K_n' \approx K_n$ where each element of the first column has the value $+1$ and each element of the first row, except the first, the value -1. We therefore assume that K_n has this special form. By adding the first column in succession to the other $n-1$ columns and expanding in terms of the first row, we find that $|K_n| = |P_{n-1}|$, where P_{n-1} is a matrix of order $n-1$, each element of which has the value 2 or 0. Therefore

$$|K_n| = |P_{n-1}| = 2^{n-1}|A_{n-1}|, \tag{1}$$

where each element of A_{n-1} has the value 1 or 0. Since A_{n-1} can be obtained from the first minor of the first element of $|K_n|$ by replacing each element -1 by 0, it is easy to reverse the process and, given A_{n-1}, write the corresponding K_n.

If A'_{n-1} can be obtained from A_{n-1} by a rearrangement of the rows of A_{n-1} or by transposing A_{n-1} or by a succession of such changes, we shall say that A_{n-1} and A'_{n-1} are equivalent, $A_{n-1} \approx A'_{n-1}$. It is evident that, if A_{n-1} and A'_{n-1} are equivalent, so are the corresponding matrices K_n and K'_n but that the converse is not true. In K_n the first row and the first column have special forms. If we wish the $(i+1)$th row of K_n to take the place of the first, each element of the $(i+1)$th row, except the first element, must be -1. This can be accomplished by multiplying each column, in which the element in the $(i+1)$th row is $+1$, by -1. The $(i+1)$th row is now the same as the original first row and the first row the same as the original $(i+1)$th row. If we now interchange the first and the $(i+1)$th rows, we have a matrix $K'_n \approx K_n$. The matrix A'_{n-1} corresponding to K'_n may therefore be obtained directly from the matrix A_{n-1} by keeping the ith row and column unchanged, by keeping any column with a 0 in the ith row unchanged, and by replacing all other elements 1 by 0 and all other elements 0 by 1. From this we obtain the following simple rule for finding A'_{n-1}. The ith row of A'_{n-1} is the same as the ith row of A_{n-1}; to find the jth row of A'_{n-1}, $j \neq i$, we add the ith row of A_{n-1} to the jth row and reduce each element modulo 2.

When A'_{n-1} can be obtained from A_{n-1} by a succession of such processes we shall say that A_{n-1} and A'_{n-1} are semi-equivalent, $A_{n-1} \sim A'_{n-1}$. While it is evident that, if A_{n-1} and A'_{n-1} are semi-equivalent, the corresponding matrices K_n and K'_n are equivalent, the following lemma is also true:

LEMMA 1. *If A_{n-2} is a submatrix of A_{n-1} and $A_{n-2} \sim A'_{n-2}$, there exists a matrix $A'_{n-1} \sim A_{n-1}$ of which A'_{n-2} is a sub-matrix.*

To find the matrix A'_{n-1} we simply apply the transformations, by which A'_{n-2} is obtained from A_{n-2}, to the matrix A_{n-1}.

By Δ_{n-1} we shall mean the determinant of a matrix A_{n-1} and by D_{n-1} the determinant of any A_{n-1} which has a maximum possible value. Our main purpose is now to prove that D_6 is 9 and that all matrices K_7 whose determinants have the value $2^6 9$ are equivalent.

3. A fundamental process. Let $A_{n-1} = (a_{ij})$ and let $D_{n-1} = |A_{n-1}|$. Then, if $a_{ij} = 1$, its cofactor $A_{ij} \geqq 0$, and, if $a_{ij} = 0$, $A_{ij} \leqq 0$, as otherwise D_{n-1} would not have a maximum value. If

$$\Delta_n = \begin{vmatrix} q & \epsilon \\ \delta' & A_{n-1} \end{vmatrix},$$

where ϵ is the row vector $(e_1, e_2, \cdots, e_{n-1})$ and δ' the transpose of the row vector $(d_1, d_2, \cdots, d_{n-1})$,

$$\Delta_n = q|A_{n-1}| - d_1 \sum_{j=1}^{n-1} e_j A_{1j} - d_2 \sum_{j=1}^{n-1} e_j A_{2j} - \cdots - d_{n-1} \sum_{j=1}^{n-1} e_j A_{n-1,j}$$

$$= qD_{n-1} - \sum_{i,j=1}^{n-1} d_i e_j A_{ij}.$$

Since each d_i or e_j has the value 1 or 0,

$$\Delta_n = qD_{n-1} - \sum' A_{ij}, \tag{2}$$

where $\sum' A_{ij}$ is the summation of the cofactors A_{ij} of all the elements of a submatrix of A. This submatrix is formed from the rows of A, for which the corresponding d_i have the value 1, and from the columns, for which the corresponding e_j have the value 1. If $q=1$, Δ_n will have its largest absolute value when $\sum' A_{ij}$ has its minimum negative value. If $q=0$, Δ_n will have its maximum absolute value when $\sum' A_{ij}$ is a maximum.

For small values of n it is easy to determine what submatrices of A to choose to make the absolute value of Δ_n a maximum. We shall now apply this process several times, starting with $n=3$.

4. Determination of D_5. It is obvious that $D_2=1$ and that there are two nonequivalent matrices A_2 and A_2' whose determinants have the value 1. They are the matrices

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since $A_2 \sim A_2'$, there is only one type of matrix K_3 with the maximum value 4 for its determinant. Any $\Delta_3 > 0$ must contain a D_2 as a first minor and in finding D_3, as a consequence of Lemma 1, we need only consider the case when $D_2 = |A_2|$. Since the cofactor of any element of D_2 is positive or zero, in (2) $q=0$, and we obtain

$$A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Therefore $D_3=2$ and there is a single type of matrix K_4 with the maximum value 16 for its determinant.

If D_4 contains a D_3 as a first minor, A_4 must contain a submatrix

$$A_3' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

with a matrix of cofactors

$$B_3 = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

The sum of the elements of any submatrix of B_3 is a minimum for the matrix b_{13} and a maximum for B_3. Therefore by (2) we obtain two matrices

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_4' = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

the absolute value of whose determinants is 3. Although A_4 is not $\approx A_4'$, $A_4 \sim A_4'$ and there is a single type of matrix K_5. If $\Delta_4 \geqq 3$ and does not contain a D_3 as a first minor, it can have no row with less than three 1's in it, and this is easily seen to be impossible. Therefore $D_4 = 3$ and there is a single type of matrix K_5 whose determinant has the maximum value 48.

To find D_5, if D_5 contains a D_4 as a first minor, by Lemma 1 we need only consider the possibility that A_5 contains a submatrix A_4'' equivalent to A_4 or A_4'. We choose

$$A_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

which has as its matrix of cofactors the matrix

$$B_4 = \begin{bmatrix} 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix}.$$

The sum of the elements of any submatrix of B_4 has a minimum value -2 for the submatrices of which b_{14} and

$$\begin{bmatrix} b_{11} & b_{14} \\ b_{21} & b_{24} \end{bmatrix}$$

are typical, and this sum has a maximum value 4. By (2) we obtain matrices

$$A_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_5' = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix},$$

whose determinants have a maximum value 5. Though A_5 is not $\approx A_5'$, $A_5 \sim A_5'$, and there is only one type of matrix K_6. (Actually there are three types of non-equivalent A_5 whose determinants have the value 5. This can be shown by considering both the matrices A_4 and A_4').

If $\Delta_5 \geqq 5$ and does not contain a first minor of absolute value 3, Δ_5 can have no row (1 1 0 0 0). Nor can it have two rows of types

$$(\alpha)\begin{bmatrix}1&1&1&0&0\\0&1&1&0&1\end{bmatrix} \quad \text{or} \quad (\beta)\begin{bmatrix}1&1&1&1&0\\0&1&1&1&1\end{bmatrix},$$

since in either case it would equal a determinant with the row (1 0 0 0 −1). Accordingly such a determinant must contain at least four rows with three 1's in each row, two of which must be of type (α). Therefore $D_5=5$ and there is a single type of matrix K_6 with the maximum value 160 for its determinant.

5. Determination of D_6. Let $\Delta_6=|A_6|$ contain a first minor which is a D_5. To obtain all non-equivalent A_6 we should have to consider all non-equivalent A_5. To find all non-equivalent K_7 we need consider only, as a consequence of Lemma 1, one A_5. We choose

$$A_5=\begin{bmatrix}0&1&1&1&1\\1&1&0&0&1\\1&0&1&0&1\\1&0&0&1&1\\1&1&1&1&0\end{bmatrix}$$

and find that the matrix of cofactors is

$$B_5=\begin{bmatrix}-3&1&1&1&2\\1&3&-2&-2&1\\1&-2&3&-2&1\\1&-2&-2&3&1\\2&1&1&1&-3\end{bmatrix}.$$

The sum of the elements of any submatrix of B_5 has a minimum value -4 for the typical submatrices

$$(b_{23}\quad b_{24}) \quad \text{and} \quad \begin{bmatrix}b_{11}&b_{13}&b_{14}\\b_{21}&b_{23}&b_{24}\end{bmatrix},$$

and a maximum value 8. Hence Δ_6 has a maximum value 9 and is obtained from

$$A_6=\begin{bmatrix}1&0&0&1&1&0\\0&0&1&1&1&1\\1&1&1&0&0&1\\0&1&0&1&0&1\\0&1&0&0&1&1\\0&1&1&1&1&0\end{bmatrix} \quad \text{and} \quad A_6'=\begin{bmatrix}1&1&0&1&1&0\\1&0&1&1&1&1\\1&1&1&0&0&1\\0&1&0&1&0&1\\0&1&0&0&1&1\\0&1&1&1&1&0\end{bmatrix}.$$

Once again A_6 is not $\approx A_6'$ but $A_6 \sim A_6'$, so that there is only one type of matrix K_7. (These are, however, actually five non-equivalent A_6 whose determinants have the maximum value 9.)

If Δ_6 does not contain a D_5 as a minor and if $\Delta_6 \geqq 9$, Δ_6 cannot contain a row with only two 1's or two rows of the types

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Moreover Δ_6 cannot contain two rows of the type

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

or three rows containing three 1's in each row with a column of three 1's.

If Δ_6 contains a row with five 1's and the last element 0, each other element in the last column must be 1. Therefore Δ_6 can only have two rows with three 1's and two rows with four 1's, and it is impossible to obtain six rows.

If Δ_6 does not contain a row with five 1's, Δ_6 may have at most three rows with four 1's of the type

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and four rows with three 1's of the type

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

There are accordingly two possibilities for Δ_6;

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{vmatrix},$$

both of which have the value ± 8. Therefore $D_6 = 9$ and *there is only one type of matrix K_7 whose determinant has the maximum value $2^6 9$.*

The fact that $D_6=9$ is interesting for two reasons. First it falls far short of the possible value given by Hadamard's Theorem. For, by this theorem, $|K_7|=m2^6<7^{7/2}$ and m could be as large as 14. Second, an Hadamard matrix H_8, *i.e.* a matrix K_8 with the maximum value 8^4 for its determinant, exists [2]. Since H_8, apart from a factor common to all its elements, is orthogonal, each first minor of $|H_8|$ has the value $\pm 8^4/8=\pm 8^3$. Therefore any first minor of $|H_8|$ determines by (1) a Δ_6 of value $8^3/2^6=8$. Accordingly *no first minor of an Hadamard determinant* $|H_8|$ *has the greatest possible value.*

6. A value for Δ_{4n}. If A is the matrix obtained by (1) from an Hadamard matrix H_{4n},

$$\Delta_{4n-1}=|A|=(4n)^{2n}/2^{4n-1}.$$

In each row of Δ_{4n-1} there are $2n$ elements 1 and $2n-1$ elements 0, while the cofactor of each element 1 is n^{2n-1} and of each element 0 is $-n^{2n-1}$. Therefore, if B is the adjugate matrix of A, the sum of all the elements of B is $(4n-1)n^{2n-1}$ and by (2) there exists a Δ_{4n} of value $(4n-1)n^{2n-1}$. Further, in any two rows of A there are exactly $n-1$ columns with both elements 0 and $2n$ columns with one element 0 and the other element 1. Therefore there exists a submatrix of B, consisting of $4n-3$ rows and $3n-1$ columns, the sum of whose elements is $[3(n-1)+2n]n^{2n-1}$. Accordingly there exists a Δ_{4n} such that

$$\Delta_{4n}=(5n-3)n^{2n-1}. \tag{3}$$

Examination of the matrix B, when $n=2$, 3, or 4, did not lead to any value of Δ_{4n} greater than that given by (3), but, when $n=5$, a Δ_{20} of value 25×5^9 was obtained.

7. Values for Δ_8, Δ_9 and Δ_{10}.
When $n=2$, we find

$$\Delta_8=\begin{vmatrix} 1&1&0&0&0&0&0&0\\ 0&1&0&0&1&0&1&1\\ 0&1&1&0&0&1&0&1\\ 0&1&1&1&0&0&1&0\\ 1&0&1&1&1&0&0&1\\ 0&1&0&1&1&1&0&0\\ 1&0&1&0&1&1&1&0\\ 1&0&0&1&0&1&1&0 \end{vmatrix},$$

which has the value 56. A calculation shows that the sum of the elements in the submatrix

$$\begin{bmatrix} b_{13}&b_{14}&b_{16}\\ b_{23}&b_{24}&b_{26} \end{bmatrix}$$

of the corresponding matrix B is -60. Therefore, if this submatrix is used to find by (2) a matrix A of order 9, $|A| = 116$. The element $a_{33} = 1$ while its cofactor has the value -4. If C is the matrix obtained from A by replacing the element a_{33} by 0, $|C| = 120$ and the sum of all of the elements in the adjugate of C is 264. Therefore the determinant

$$\Delta_{10} = \begin{vmatrix} 0 & x \\ x' & C \end{vmatrix},$$

where x denotes a row of 9 1's and x' a column of 9 1's, has the value 264. We have thus obtained determinants Δ_8, Δ_9 and Δ_{10} with the values 56, 120 and 264 respectively. These numbers fall far short of the corresponding maximum ones given by Hadamard's Theorem, which are 76, 195 and 521, but as D_5 is 5 and not 6 and D_6 is 9 and not 14, we already know that the maxima given by Hadamard's Theorem are not always attained.

Since $(4n)^{-1/2}H_{4n}$ is an orthogonal matrix and any minor of an orthogonal matrix is equal in absolute value to its complementary minor,

$$\Delta_{4n-m-1} = n^{2n-m}\Delta_{m-1}, \tag{4}$$

where Δ_j is obtained by (1) from a minor of order $j+1$ of H_{4n}. The maximum values of Δ_j obtained from minors of $|H_{12}|$ are therefore $\Delta_{10} = 243$, $\Delta_9 = 81$, $\Delta_8 = 27$, $\Delta_7 = 18$ and $\Delta_6 = 9$ and these values, except the last, are all less than those previously obtained. Accordingly $|H_{12}|$ *contains a minor of order* 7 *with the maximum value* $2^6 9$ *but does not contain a minor of order greater than* 7 *with the maximum value.*

In fact, if $n > 2$, the value n^{2n-3} for Δ_{4n-4} in (4) is less than the value $(5n-8)(n-1)^{2n-3}$ given by (3) and the value $2n^{2n-4}$ for Δ_{4n-5} much less than $D_{4n-5} = 2(n-1)^{2n-2}$. These last comparisons are based on the assumption that Hadamard matrices of orders $4n$ and $4n-4$ both exist. Unfortunately, it is not known at present whether or not this is true [3].

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Theft. Charlie was crazy about mathematics. He could do almost any example in his head. He said he would like to go to college, if he could get in, so he could learn more mathematics. But he didn't get in any college. He got into a bank in Dover Plains, and after he'd worked there twenty or thirty years he got in jail. Well, I might have ended in jail, too, like Charlie, but I could never be sure if it was πr^2 or $r\pi^2$, and now I've forgotten what π is—or maybe its r, and I guess I'm just as well off. Percival Wilde, *Inquest.*—*E. D. Schell.*

HELMERT'S DISTRIBUTION

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1. Purpose. The joint distribution of sample mean and standard deviation for samples of independent random observations, drawn from a normal population, may be called *Helmert's distribution.** The purpose of this note is to derive Helmert's distribution by a straightforward inductive procedure.

2. Background. Helmert's distribution is of central importance in the theory of sampling from a normal (*i.e.* Gaussian) population because from it follow:

1. The statistical independence of sample mean and sample standard deviation.
2. The separate single distributions of sample mean and sample standard deviation.
3. The distribution of the Student t.

Helmert's distribution was apparently first established (in effect) by Helmert in reference [1]; perhaps a more readily accessible version of his proof appears on pages 151 to 163 of reference [2]. Helmert's procedure was to set up a pair of linear transformations which transformed the joint distribution of the individual errors of observation into a joint distribution of sample mean error and standard deviation, plus dummy variables which could be "integrated out" over all possible values.

Helmert's distribution may also be established by an appeal to the geometry of hyper-space, as in reference [3] and similarly in reference [4], pages 129 to 133. A third method of proof is to first establish a theorem about the independent chi-square distributions of quadratic forms, under suitable restrictions. Reference [5], pages 105 to 110, follows this procedure, first proving Cochran's theorem and then showing that Helmert's distribution is an immediate consequence.

It seems of pedagogical and conceptual interest that Helmert's distribution may be simply and directly established by mathematical induction. Although Helmert himself used an inductive method in reference [6] to establish a somewhat simpler distribution, it is apparently not generally recognized that the joint distribution with which we are concerned may be inductively derived.

3. Notation. Notation will be similar to that of reference (3). We suppose that n independent random observations, x_i, have been drawn from a normal population with mean μ and standard deviation σ.

Sample mean: $\bar{x} = (\sum x_i)/n$

Error of sample mean: $u = \bar{x} - \mu$

Error of individual observation: $\epsilon_i = x_i - \mu$

Sample standard deviation squared (sample variance):

$$s^2 = \frac{1}{n}\left[\sum (x_i - \bar{x})^2\right] = \frac{1}{n}\left[\left(\sum \epsilon_i^2\right) - nu^2\right].$$

* This name is sometimes given to the single distribution of sample standard deviation.

4. Statement of Helmert's distribution. Helmert's distribution is given by the distribution element:

$$(1) \qquad K_1 s^{n-2} \exp\left[-\frac{n}{2\sigma^2}(s^2+u^2)\right] du\,ds$$

$$K_1 = \frac{n^{n/2}}{2^{(n-2)/2}\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)\sigma^n}.$$

That is, the probability that u and s will simultaneously lie within given limits is the double integral, over those limits, of (1).

5. Proof. Suppose that (1) is the correct probability element for a sample of size n. Increase the sample size to $n+1$ by drawing a new independent observation, x_{n+1}, from the parent normal population. The corresponding ϵ_{n+1} will of course be distributed normally with zero mean and standard deviation σ. Hence the probability element for the joint distribution of u, s, and ϵ_{n+1} will be:

$$(2) \qquad \frac{K_1}{\sigma\sqrt{2\pi}} s^{n-2} \exp\left[-\frac{n}{2\sigma^2}(s^2+u^2+\epsilon_{n+1}^2/n)\right] du\,ds\,d\epsilon_{n+1}.$$

Let $\bar{u}$ and $\bar{s}$ be, respectively, the error of the new sample's mean and the new sample's standard deviation. The plan of proof is as follows:

1. Express $\bar{u}$ and $\bar{s}$ in terms of u, s, and ϵ_{n+1}.
2. Transform the distribution element (2) into the distribution element for $\bar{u}$, $\bar{s}$, and ϵ_{n+1}.
3. Integrate the resulting transformed distribution element over all possible values of ϵ_{n+1}.

The distribution element thus obtained for $\bar{u}$ and $\bar{s}$ will have the same form as (1) except that "n" will be replaced by "$n+1$." Finally Helmert's distribution will be established for the special case, $n=2$.

The relevant relationships between the five variables, u, s, $\bar{u}$, $\bar{s}$, and ϵ_{n+1}, may readily be obtained from their definitions:

$$\begin{cases} u = \dfrac{1}{n}\left[(n+1)\bar{u} - \epsilon_{n+1}\right] \\ s^2 = \dfrac{n+1}{n^2}\left[n\bar{s}^2 - (\bar{u} - \epsilon_{n+1})^2\right] \end{cases}$$

$$\begin{cases} \bar{u} = \dfrac{1}{n+1}\left[nu + \epsilon_{n+1}\right] \\ \bar{s}^2 = \dfrac{n}{(n+1)^2}\left[(n+1)s^2 + (u - \epsilon_{n+1})^2\right] \end{cases}$$

$$n(s^2 + u^2 + \epsilon_{n+1}^2/n) = (n+1)(\bar{s}^2 + \bar{u}^2).$$

Henceforth the subscript will be dropped from "ϵ_{n+1}". The Jacobian of the above transformation is:

$$\frac{\partial u}{\partial \bar{u}} \frac{\partial s}{\partial \bar{s}} = \frac{(n+1)^{3/2}}{n} \bar{s}\left[n\bar{s}^2 - (\bar{u} - \epsilon)^2\right]^{-1/2}.$$

Hence the joint distribution element for $\bar{u}$, $\bar{s}$, and ϵ is:

$$K_2\bar{s}\left[n\bar{s}^2 - (\bar{u} - \epsilon)^2\right]^{(n-3)/2} \exp\left[-\frac{n+1}{2\sigma^2}(\bar{s}^2 + \bar{u}^2)\right] d\bar{u}d\bar{s}d\epsilon$$

$$K_2 = \frac{K_1}{\sigma\sqrt{2\pi}} \frac{(n+1)^{(n+1)/2}}{n^{n-1}}.$$

We now wish to integrate this distribution element over all (real) values of ϵ. Since $s^2 \geqq 0$, ϵ is limited by the inequality:

$$\epsilon^2 - 2\bar{u}\epsilon - n\bar{s}^2 + \bar{u}^2 \leqq 0. \tag{3}$$

(3) is satisfied if and only if ϵ lies between $\bar{u} - \bar{s}\sqrt{n}$ and $\bar{u} + \bar{s}\sqrt{n}$. Hence it is necessary to evaluate:

$$\int_{\bar{u}-\bar{s}\sqrt{n}}^{\bar{u}+\bar{s}\sqrt{n}} \left[n\bar{s}^2 - (\bar{u} - \epsilon)^2\right]^{(n-3)/2} d\epsilon = 2n^{(n-2)/2}\bar{s}^{n-2} \int_0^1 (1 - y^2)^{(n-3)/2} dy$$

by symmetry of the integrand, putting $\epsilon = \bar{u} - \sqrt{n}\bar{s}y$. Now let $y = \sin\theta$. The integral becomes:

$$2n^{(n-2)/2}\bar{s}^{n-2} \int_0^{\pi/2} \cos^{n-2}\theta d\theta.$$

But this is a well known special case of the complete Beta integral. (See, for example, formula 483 of reference [7].) The integral is equal to:

$$n^{(n-2)/2}\bar{s}^{n-2}\sqrt{\pi} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

Therefore the final distribution element is:

$$K_3\bar{s}^{n-1} \exp\left[-\frac{n+1}{2\sigma^2}(\bar{s}^2 + \bar{u}^2)\right] d\bar{u}d\bar{s}$$

$$K_3 = \frac{(n+1)^{(n+1)/2}}{2^{(n-1)/2}\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)\sigma^{n+1}}$$

and this is just (1) with "n" replaced by "$n+1$."

It remains to show that Helmert's distribution holds for the case $n=2$. Here the joint distribution element for ϵ_1 and ϵ_2 is:

$$\frac{1}{2\pi\sigma^2}\exp\left(-\frac{\epsilon_1^2+\epsilon_2^2}{2\sigma^2}\right)d\epsilon_1 d\epsilon_2.$$

Make the transformation:

$$\left\{\begin{aligned}\epsilon_1 &= u+l\\ \epsilon_2 &= u-l\end{aligned}\right\} \qquad \left\{\begin{aligned}u &= \tfrac{1}{2}(\epsilon_1+\epsilon_2)\\ l &= \tfrac{1}{2}(\epsilon_1-\epsilon_2)\end{aligned}\right\}$$

where u is the sample mean error and l is a variable whose absolute value is the sample standard deviation. The absolute value of the Jacobian is 2, and:

$$\epsilon_1^2+\epsilon_2^2 = 2u^2+2l^2.$$

Hence the joint distribution element for u and l is:

$$\frac{1}{\pi\sigma^2}\exp\left[-\frac{u^2+l^2}{\sigma^2}\right]dudl.$$

And since the sample standard deviation, s, is the absolute value of l, the joint distribution element for u and s is:

$$\frac{2}{\pi\sigma^2}\exp\left[-\frac{u^2+s^2}{\sigma^2}\right]duds$$

which is just (1) with $n=2$.

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Woolgathering. While studying the *pons asinorum* in Euclid, he suffered every cuddie upon the common to trespass upon a large field belonging to the Laird. Walter Scott, *Heart of Mid-Lothian.*—*E. D. Schell*

PARTIALLY ORDERED FIELDS AND GEOMETRIES*

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1. Introduction. Despite its simplicity, *order* is one of the most important and pervasive ideas in modern mathematics. It is everpresent in analysis from the elementary process of counting to the refined techniques of modern function theory and plays an important, if not quite so obvious a role in algebra and geometry.

The abstract theory of order usually is based on a 2-term relation $a<b$ (read "a precedes b") called *precedence*, which satisfies the following postulates, where a, b, c denote arbitrary elements of a given set S.

P1. *$a<b$ implies that a, b are distinct.*

P2. *$a<b, b<c$ imply $a<c$.*

P3. *If a, b are distinct, one of the relations $a<b, b<a$ holds.*†

On the other hand an equivalent theory of order may be developed in terms of a 3-term relation *betweenness*, denoted (abc), (read "b is between a and c" or "a, b, c are in the order abc") which satisfies the following postulates:

B1. *(abc) implies that a, b, c are distinct.*

B2. *(abc) implies (cba).*

B3. *$(abc), (bcd)$ imply $(abd), (acd)$.*

B4. *$(abc), (acd)$ imply $(abd), (bcd)$.*

B5. *If a, b, c are distinct, one of the relations $(abc), (bca), (cab)$ holds.*‡

Observe that P3 and B5 play similar roles in the respective theories, requiring that two or three elements be in a sense "comparable."

To show the two theories of order equivalent, we introduce in a system satisfying P1, P2, P3

DEFINITION A. If elements a, b, c satisfy the conditions $a<b, b<c$ or $c<b, b<a$ we say (abc).

It is easily seen that the 3-term relation so defined satisfies B1, · · · , B5. Thus any "precedence" system satisfying P1, P2, P3 is converted by Definition A into a "betweenness" system satisfying B1, · · · , B5. Conversely every abstract betweenness system satisfying B1, · · · , B5 arises from some precedence system satisfying P1, P2, P3 in just this way.§ Thus the two theories of order are logically equivalent and it is immaterial in theory which we choose to employ. In algebra precedence is preferred because of its simplicity, while in geometry it seems more natural to use betweenness relations.

* The author wishes to express his thanks to his former student Frank Harary for working out details of several of the proofs.

† See J. W. Young, Lectures on Fundamental Concepts of Algebra and Geometry, New York, 1911, p. 68.

‡ See E. V. Huntington and J. R. Kline, Sets of independent postulates for betweenness, Trans. Amer. Math. Soc. 18, 1917, pp. 301–325 for a very detailed study of several postulate systems for betweenness and their independence. Our choice of postulates was made for simplicity and convenience rather than irredundance.

§ See Huntington and Kline, *loc. cit.* p. 324 for the definition of the associated precedence relation.

Order is important in algebra not merely because so many algebraic systems are ordered, often in a unique or so-called "natural" manner, but also because ordinal notions and algebraic ideas often are closely interlinked. Thus consider a field F in which is defined a relation $<$ satisfying P1, P2, P3. Suppose the order relation $<$ is linked to the field operations $+$, $\times$ by the monotonic laws

M1. $a<b$ *implies* $a+x<b+x$,

M2. $a<b$, $0<x$ *imply* $ax<bx$.

These conditions characterize F as an *ordered field* of which the rational number system is the simplest example and the real number system the most important.*

In geometry the most important example of order is the relation betweenness for points on a line. We observe that this is closely related to the theory of ordered fields since if we set up an analytic geometry in an ordered field in the usual manner and define between in the natural way, postulates B1, $\cdots$, B5 are satisfied for the points of a line.

Recently there has been developed a flourishing generalization of the classical theory of order, based merely on P1, P2, called the theory of *partial* order. Since P3, the postulate of "comparability" has been omitted, this theory is applicable to what might be termed non-linear situations, and has many interesting and important applications to relations such as set inclusion, divisibility, implication arising in algebra, geometry, analysis and logic.†

As we have noted, B5 plays the same role in the theory of betweenness as P3 in the theory of precedence. This suggests a theory of 3-term partial order based merely on B1, $\cdots$, B4. One would expect such a theory to be equivalent to the theory of 2-term partial order in the sense in which the original theories of precedence and betweenness are equivalent—namely with Definition A forming the connecting link.

This is not so. For there exist 3-term relations satisfying B1, $\cdots$, B4 which can not be generated by Definition A from 2-term relations satisfying P1, P2. As an illustration consider (pqr) the betweenness relation for points in the euclidean *plane*, which requires that q be an interior point of segment pr. Clearly the relation satisfies B1, $\cdots$, B4 but not B5. Suppose this relation definable in terms of some relation $<$ satisfying P1, P2 by Definition A. Let a, b be distinct points. There exists a point x such that (axb). Hence by Definition A $a<x$, $x<b$ or $b<x$, $x<a$ so that P2 implies $a<b$ or $b<a$. Thus P3 holds. Hence $<$ is a "precedence" relation and the given relation *between* must satisfy B5. This contradiction justifies our assertion.

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* For equivalent characterizations in terms of the notion "positive element" see (1) A. A. Albert, Modern Higher Algebra, Chicago, 1937, p. 110, (2) G. Birkhoff and S. MacLane, A Survey of Modern Algebra, New York, 1941, p. 54, The monotonic laws are discussed in Young, *loc. cit.*, pp. 100, 110–111.

† See Garrett Birkhoff, Lattice Theory, New York, 1940, Chap. I, and Lattice theory, this Monthly, 50, 1943, pp. 484–487. Our postulates P1, P2, are equivalent to P1, P2, P3 of Birkhoff, see Lemma 2, p. 7 of the first reference.

Since this theory of 3-term partial order is not equivalent to the theory of 2-term partial order in the simple and natural sense described above, it seems worthy of development on its own account. In this note we construct two elementary applications of this idea, first to a generalization of ordered field which we term *partially ordered field*, second to the notion *partially ordered geometry*, a generalization of ordinary linear geometry in which no one of three points on a line need be between the other two.*

2. Partially ordered fields. We begin by stating the precise sense in which we shall employ the term partial ordering.

Consider a set S of elements $a, b, c, \cdots$ in which is defined a 3-term relation called *order*, indicated by the notation (abc). We say S is a *partially ordered* set, or S is *partially ordered* by the relation, or the relation is a *partial ordering* of S if postulates B1, $\cdots$, B4 and the following existential postulate B6 are satisfied:

B6. *There exist elements a, b, c such that (abc).*

If in addition postulate B5 is satisfied (classic theory of betweenness), we say S is *fully* or *simply ordered*, or the relation is a *simple ordering* of S.

We observe that B6 is an implicate of B5 if S contains three or more elements; it is introduced to exclude cases in which the order relation is vacuous. Although we can easily weaken the postulate system B1, $\cdots$, B4, B6 we shall not do so, since it is not our object to study the logical interrelations of the postulates but rather to present a simple and natural generalization of the familiar theory of order.

Before continuing with partially ordered *fields* we give one very elementary property of partially ordered sets.

THEOREM. *In a partially ordered set, (abc) implies (bca) false.*

Proof. By B3, (abc) and (bca) imply (aba) contrary to B1.

Let $\mathfrak{F}$ be a field which is partially (simply) ordered by a relation (abc). If $\mathfrak{F}$ satisfies the following *invariance principles* I1, I2, we say it is a *partially (simply) ordered field*:

I1. (abc) *implies* $(a+x\ b+x\ c+x)$;

I2. (abc) *implies* $(ax\ bx\ cx)$ *provided* $x \neq 0$.

These principles correspond to the monotonic laws M1, M2 for 2-term order but are neater and more symmetrical in form. The notion partially ordered field is a generalization of ordered field as characterized in §1, since the latter can be shown to be equivalent to simply ordered field as we have defined it.

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* That is B5 need not hold for points of a line. The present paper had its inception in the attempt to show the independence of one of the postulates (equivalent to assuming B5 for collinear points) in a new type of characterization of ordered linear geometries given in Prenowitz, Descriptive geometries as multigroups, Trans. Amer. Math. Soc., vol. 59, 1946, pp. 333–380.

3. Elementary properties. We now deduce some elementary formal properties of partially ordered fields which we shall employ later.

A. $(0a1)$ *implies* $(0\ 1-a\ 1)$.

Proof. $(0a1)$ implies $(0\ -a\ -1)$ by I2. From this by I1 we have $(1\ 1-a\ 0)$ so that $(0\ 1-a\ 1)$ follows by B2.

B. $(0a1)$, $(0b1)$ *imply* $(0\ ab\ 1)$.

Proof. By I2, $(0a1)$ implies $(0\ ab\ b)$. This with $(0b1)$ implies $(0\ ab\ 1)$ by B4.

C. $(0a1)$ *implies* $(0\ a\ 1+a)$, $(0\ 1\ 1+a)$.

Proof. By Property A, $(0a1)$ implies $(0\ 1-a\ 1)$. By I1 this yields $(a\ 1\ 1+a)$ which with $(0a1)$ implies the desired conclusions by B3.

D. $(0a1)$, $(0b1)$ *imply* $(0\ a\ a+b)$.

Proof. Adding a to each term in the relation $(0b1)$ and applying B2 we get

$$(1 + a\ a + b\ a). \tag{1}$$

By Property C and B2 $(0a1)$ implies

$$(1 + a\ a\ 0). \tag{2}$$

Applying B4 to (1), (2) we get $(a+b\ a\ 0)$, and $(0\ a\ a+b)$ follows by B2.

E. (acb) *is equivalent to the existence of elements* λ, μ *satisfying*

$$c = \lambda a + \mu b, \qquad \lambda + \mu = 1, \qquad (0\lambda 1), \qquad (0\mu 1). \tag{3}$$

Proof. Assume (acb). We have the algebraic identities

$$c = \lambda a + \mu b, \qquad \lambda + \mu = 1$$

where

$$\lambda = (c - b)/(a - b), \qquad \mu = (c - a)/(b - a).$$

Subtracting b from the terms in (acb) we get $(a-b\ c-b\ 0)$. Dividing in this relation by $a-b$ and applying B2 we get $(0\lambda 1)$. Similarly we show $(0\mu 1)$ and the necessity is established.

Conversely assume (3). Solving for λ we get $\lambda=(c-b)/(a-b)$. Hence multiplying in $(0\lambda 1)$ by $a-b$ we get $(0\ c-b\ a-b)$. Adding b to the terms in this relation and applying B2 we get (acb), which completes the proof.

4. The characteristic. We show now that the characteristic of a partially ordered field, as in the familiar theory of (simply) ordered fields, is infinite.

Let $\mathfrak{F}$ be a partially ordered field. By B6 there exist elements a, b, c, in $\mathfrak{F}$ satisfying (abc). Subtracting a from the terms in this relation and dividing by $c-a$ we get

$$(0x1) \tag{1}$$

where $x=(b-a)/(c-a)$. By Property C, (1) implies

$$(0\ 1\ 1 + x). \tag{2}$$

Adding 1 to the terms in (1) we get

$$(1\ 1 + x\ 2) \tag{3}$$

where $2=1+1$. By B3 relations (2), (3) imply

$$(012).$$

Repeatedly adding 1 to the terms in this relation we get

$$(123), (234), \cdots, (n\ n+1\ n+2)$$

where $3=2+1$, $4=3+1$ and n is an arbitrary element of $\mathfrak{J}$ obtained by summation of 1. From this it follows by B3 that the elements $1, 2, 3, \cdots, n$ are ordered in the *natural* way and so are distinct. Thus $\mathfrak{J}$ has infinite characteristic. It readily follows that the rational subfield of $\mathfrak{J}$ is ordered in the natural manner. Thus we may assert the

THEOREM. *Any partially ordered field has characteristic infinity. Its rational subfield has the natural (simple) ordering.**

Applying this to the field of rational numbers we have the

COROLLARY. *The only partial ordering of the field of rational numbers is the natural ordering.*†

5. A partially ordered field. Although there is no partial ordering of the field of rational numbers which is non-simple, we can easily define non-simple orderings for more complicated fields. Let $\mathfrak{R}$ denote the real number field and $\mathfrak{R}'$ the field of rational functions in a real variable x with coefficients in $\mathfrak{R}$. We can consider the elements of $\mathfrak{R}'$ to be fractions with polynomial numerator and denominator, assuming the usual agreements for equality or equivalence of fractions. Elements of $\mathfrak{R}'$ are denoted $A, B, C, \cdots$ and their respective functional values for argument x are denoted $A(x), B(x), C(x) \cdots$. We assume the familiar simple ordering of field $\mathfrak{R}$ and indicate it by the notation (abc), which is equivalent to $a<b<c$ or $a>b>c$. Using it we induce a partial ordering of field $\mathfrak{R}'$ in the following manner.

We define (ABC) to mean that

$$(A(x)\ B(x)\ C(x))$$

holds for all x in $\mathfrak{R}$ except a finite number, or as we shall say, for *almost all* x.

Observe that this property is independent of the form in which the elements of $\mathfrak{R}'$ are expressed. Thus if we replace A, B or C by equal or equivalent elements of $\mathfrak{R}'$ the relation still holds.

It is easy to verify that $\mathfrak{R}'$ with order as defined is a partially ordered field. For example consider B3. Suppose (ABC), (BCD).‡ By definition $(A(x)\ B(x)\ C(x))$ and $(B(x)\ C(x)\ D(x))$ hold separately for almost all x. Hence they hold *simultaneously* for almost all x. Since B3 holds in $\mathfrak{R}$ we get

* Compare Albert, *loc. cit.*, pp. 110–112, Birkhoff and MacLane, *loc. cit.*, p. 56 Theorem 13.

† Compare Albert, *loc. cit.*, p. 112 Theorem 12.

‡ More strictly we should assume (ABC), $(B'C'D)$ where B', C', are equivalent to B, C respectively. But this reduces to the case given, since we can substitute B, C for B', C' by the observation in the preceding paragraph.

$$(A(x)\ B(x)\ D(x)), \qquad (A(x)\ C(x)\ D(x))$$

for almost all x and B3 is verified in $\mathfrak{R}'$.

Observe that the elements of $\mathfrak{R}$, which are included in $\mathfrak{R}'$ as "constant" functions, maintain in $\mathfrak{R}'$ their familiar order in $\mathfrak{R}$. Thus we may say that the ordering in $\mathfrak{R}'$ is an *extension* of that in $\mathfrak{R}$. Moreover $\mathfrak{R}'$ is not simply ordered. For there subsists no order relation involving the function x and the constant functions 0, 1, so that B5 fails. Thus we have constructed a non-simple ordering in $\mathfrak{R}'$ which is an extension of the familiar simple ordering in $\mathfrak{R}$. This construction can be applied to any partially ordered field $\mathfrak{F}$ to yield a non-simply ordered field $\mathfrak{F}'$, the ordering in which is an extension of that in $\mathfrak{F}$.

6. Linearity. In this section we clarify the notion *linear* or *one-dimensional* ordering and show that $\mathfrak{R}'$ is linearly ordered.

All simply ordered sets are in a natural sense linear or one-dimensional. This is not so for partially ordered sets. For example a euclidean plane geometry is a partially ordered set and we would say it has a two-dimensional rather than a linear type of order. Thus it is important to clarify the notion *linear ordering* as one of the simplest types of partial ordering.

We define linear ordering as a natural generalization of simple ordering. We say that partially ordered set S is *linearly ordered* or is *linear* provided each element of S is related to any two given elements of S by a finite chain of order relations. We express this in more formal language. Let us say in a partially ordered set S that c is *linearly related* to a, b where $a \neq b$, provided there exists in S a sequence

$$x_1 = a, \qquad x_2 = b, \cdots, x_n = c$$

such that for each i, $3 \leqq i \leqq n$, there exist j, $k < i$ for which one of the relations

$$(x_i x_j x_k), \qquad (x_j x_i x_k)$$

holds. Then, if each element of S is linearly related to each pair of elements of S, we say S is *linearly ordered* or is *linear*. A simple ordering is the most elementary example of a linear ordering since in this case n has the minimum possible value 3, if $c \neq a, b$.

We can now state the

THEOREM. *The field $\mathfrak{R}'$ is linearly ordered.*

Proof. Let A, B, C be elements of $\mathfrak{R}'$, where $A \neq B$. We shall show that C is linearly related to A, B.

First we consider the case $A = 0$, $B = 1$. Assume that C is a polynomial and express $C(x)$ in the form

$$C(x) = c_0 + c_1 x + \cdots + c_n x^n$$

where n is even, so that $c_n = 0$ is possible. We determine C' by the relation

$$C'(x) = 2 + \sum_{i=0}^{n} |c_i| + (|c_1| + |c_2|)x^2 + (|c_3| + |c_4|)x^4 + \cdots + (|c_{n-1}| + |c_n|)x^n.$$

Since

$$1 < C'(x), \qquad C(x) < C'(x)$$

for all x, we have

$$(0\ 1\ C'(x)), \qquad (0\ 1\ C'(x) + 1), \qquad (C(x)\ C'(x)\ C'(x) + 1)$$

for all x. Thus any polynomial C is linearly related to 0, 1, the sequence 0, 1, C', $C'+1$, C establishing the desired relation.

Now we show that the reciprocal of a non-zero polynomial is linearly related to 0, 1. Suppose polynomial $C \neq 0$. We have

$$(1/C(x)\ 0\ -C(x))$$

for almost all x. Thus we have

$$(1/C\ 0\ -C). \tag{1}$$

Since C is linearly related to 0,1 as we have just shown, $-C$ is linearly related to 0, -1. Hence in view of $(-1\ 0\ 1)$, $-C$ is linearly related to 0,1. Hence (1) implies that $1/C$ is linearly related to 0,1.

Now let C be an unrestricted element of $\mathfrak{R}'$ and express C as $C_1 \cdot C_2$ where C_1 is a polynomial and C_2 is the reciprocal of a polynomial. C_1 is linearly related to 0, 1. Hence by I2, $C = C_1 \cdot C_2$ is linearly related to 0, C_2. Hence, since C_2 is linearly related to 0, 1, C is likewise. Thus the theorem is established for $A = 0$, $B = 1$.

To complete the proof we observe that the assertion C is linearly related to A, B is equivalent by I1, I2 to the assertion $(C-A)/(B-A)$ is linearly related to 0, 1.

7. Partially ordered geometries. We now wish to clarify the idea *partially ordered geometry*, *i.e.* a generalization of the familiar idea of ordered linear geometry (*e.g.* euclidean, hyperbolic or ordered affine geometry) in which the points on a line are not simply ordered. We are confronted by a difficulty at the outset: How shall we define *line*?

In the familiar theory line ab is defined as the set of points consisting of the distinct points a, b and all points which bear an order relation to a, b, *i.e.* all points x satisfying one of the relations (xab), (axb), (abx). This definition is unsatisfactory for our purpose having been designed for a theory in which a line is a *simply* ordered set. However a line in the familiar theory does have the property of containing all points bearing an order relation to any two of its points. Thus it contains with the distinct points a, b all points *linearly related* to a, b. This suggests the formal definition:

In a partially ordered system, the set of all elements linearly related to the distinct elements a, b is called a *line* and is denoted *line* ab or simply ab.

Now we define a *partially ordered geometry* as a partially ordered set S of elements called *points* which satisfies the following postulates:

G1. (Closure) *If* $a \neq b$ *there exist* x, y *in* S *satisfying* (axb), (aby).

G2. (Triangle Property) *If* a, b, c, *do not colline and* (bcx), (cya) *there exists* z *in* S *satisfying* (azb), (xyz).

G3. (Linearity) *There exists not more than one line containing two given points.*

S is called a *simply ordered geometry* if in addition it satisfies postulate

G4. *A line is simply ordered.*

This is our formulation of the familiar concept of ar *ordered linear geometry* which is referred to in the foundations of geometry as a *descriptive geometry*.* It is worthy of note that in the familiar formulation of this idea only G1, G2, G3 (or their equivalent) are assumed, but our definition of line is replaced by the more restrictive definition of line mentioned at the beginning of this section.

We now deduce two elementary properties of partially ordered geometries. First we supplement G3 by the

THEOREM. *There is at least one line containing two points* a, b *namely, line* ab.

Proof. If $a \neq b$, G1 implies (axb). Hence the existence of the sequence a, b, x, a implies that a is linearly related to a, b. Thus line ab contains a, and by symmetry it contains b.†

Now we consider the nature of the ordering of a line in a partially ordered geometry.

THEOREM. *A line is linearly ordered.*

Proof. Let p, q, r be points of ab, where $p \neq q$. We must show that r is linearly related to p, q. By the last theorem and G3 we have $ab = pq$. Hence r is contained in pq. Thus r is linearly related to p, q and the proof is complete.

This property is easily shown to be equivalent to G3 and justifies our description of G3 as a *linearity* property.

8. A partially ordered geometry. We now make the ideas of the last section concrete by constructing a partially ordered geometry which is not a simply ordered geometry.

Consider the partially ordered field $\mathfrak{R}'$ of §5. We construct an analytic geometry in $\mathfrak{R}'$, which for simplicity we make two dimensional. Let S be the set of ordered pairs (x_1, x_2) of elements of $\mathfrak{R}'$. We denote (x_1, x_2) by x and use the Greek letters α, β, λ, μ for elements of $\mathfrak{R}'$. It is convenient to introduce the vectorial notation for addition of elements of S and multiplication of elements of S by those of $\mathfrak{R}'$:

* See O. Veblen, Foundations of Geometry, in Monographs on Topics of Modern Mathematics, edited by J. W. A. Young, New York, 1915; also see H. S. M. Coxeter, Non-Euclidean Geometry, Toronto, 1942, Chap. 8.

† From this it follows that line ab is the smallest set containing the distinct points a, b and all points bearing an order relation to any two of its points. This is an interesting type of closure property well known in modern algebra. Compare, for example, "subgroup generated by a set of elements."

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2),$$
$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2),$$

where α is in $\mathfrak{R}'$.

Now we define an order relation in S: Let a, b, c be elements of S, where $a \neq b$, such that there exist λ, μ in $\mathfrak{R}'$ which satisfy

$$c = \lambda a + \mu b,^* \qquad \lambda + \mu = 1, \qquad (0\lambda 1), \qquad (0\mu 1).$$

Then we say (acb).

This definition is of course motivated by the familiar "point of division" formula of elementary cartesian geometry, where μ/λ is the ratio in which point b divides segment ac. It is also suggested by Property E of §3. We observe that the condition $(0\mu 1)$ is redundant in view of Property A of §3, but is included for symmetry of expression.

S, with order as defined, is a partially ordered set. B1, B2, B6 are easily verified. To establish B3 suppose (abc), (bcd). By definition we have

$$(1) \qquad b = \lambda a + \mu c, \qquad c = \lambda' b + \mu' d,$$

$$(2) \qquad \lambda + \mu = \lambda' + \mu' = 1, \qquad (0\lambda 1), \qquad (0\mu 1), \qquad (0\lambda' 1), \qquad (0\mu' 1).$$

Eliminating c in (1) we get

$$(3) \qquad (1 - \mu\lambda')b = \lambda a + \mu\mu' d.$$

We show $1 - \mu\lambda' \neq 0$. By Property B of §3 $(0\,\mu\mu'\,1)$. Hence by Property D we have

$$(4) \qquad (0\,\lambda\,\lambda + \mu\mu'),$$

so that

$$(5) \qquad 1 - \mu\lambda' = 1 - \mu(1 - \mu') = \lambda + \mu\mu' \neq 0$$

follows. Thus we may solve (3) for b getting

$$(6) \qquad b = \alpha a + \beta d$$

where α, β are determined by

$$(7) \qquad \alpha = \lambda/(\lambda + \mu\mu'), \qquad \beta = \mu\mu'/(\lambda + \mu\mu').$$

Clearly $\alpha + \beta = 1$ and (4) implies $(0\alpha 1)$. The supposition $a = d$ implies by (6) $b = d$, contrary to (bcd). Hence $a \neq d$ and we may assert (abd). Similarly (acd) holds. Thus B3 is verified in S. An analogous proof holds for B4. Thus S is a partially ordered set.

We now show that S is a partially ordered *geometry*. G1 is easily verified. To show G2 valid suppose a, b, c do not colline and (bcx), (cya). We seek z satisfying (azb), (xyz). We have

$$(8) \qquad c = \lambda' x + \mu' b, \qquad y = \lambda a + \mu c,$$

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* This is really shorthand for $c_i = \lambda a_i + \mu b_i$—the vector notation permits us to drop subscripts.

where the coefficients satisfy (2). Eliminating c in (8) we have

$$y = \mu\lambda' x + (\lambda a + \mu\mu' b). \tag{9}$$

Let z be determined by

$$(1 - \mu\lambda')z = \lambda a + \mu\mu' b. \tag{10}$$

Since (5) holds as in the verification of B3, z is well defined and

$$z = \alpha a + \beta b$$

where α, β are determined by (7). The conclusion (azb) easily follows, $a \neq b$ being precluded by the non-collinearity of a, b, c. Eliminating $\lambda a + \mu\mu' b$ between (9), (10) we get

$$y = \mu\lambda' x + (1 - \mu\lambda')z.$$

The supposition $x = z$ implies $x = y$. Hence (bcx), (cya) imply by B3 (bca), contrary to the non-collinearity of a, b, c. Hence $x \neq z$ and (xyz) easily follows. This completes the verification of G2.

Before considering G3 we derive the

LEMMA. Let

$$x_i = \lambda_i a + \mu_i b, \qquad \lambda_i + \mu_i = 1, \qquad (i = 1, 2, 3)$$

where a, b are distinct elements of S. Then $(x_1x_2x_3)$ if and only if $(\lambda_1\lambda_2\lambda_3)$.

Proof. Suppose $(\lambda_1\lambda_2\lambda_3)$. By Property E of §3, there exist α, β in $\mathfrak{R}'$ such that

$$\lambda_2 = \alpha\lambda_1 + \beta\lambda_3, \qquad \alpha + \beta = 1, \qquad (0\alpha 1), \qquad (0\beta 1).$$

Hence

$$\begin{aligned} x_2 &= \lambda_2 a + \mu_2 b = \lambda_2(a - b) + b = (\alpha\lambda_1 + \beta\lambda_3)(a - b) + b \\ &= \alpha(\lambda_1(a - b) + b) + \beta(\lambda_3(a - b) + b) = \alpha x_1 + \beta x_3. \end{aligned}$$

It is easily seen that $x_1 \neq x_3$, so that $(x_1x_2x_3)$ holds. The converse is proved similarly.

To verify G3 we must identify *line* in S. This we do in the following

THEOREM. *Line ab, where $a \neq b$, is the set of all x in the form*

$$x = \lambda a + \mu b, \qquad \lambda + \mu = 1, \tag{11}$$

where λ, μ are variables in $\mathfrak{R}'$.

Proof. Suppose x is in line ab. By definition, x is linearly related to a, b. Hence there exists a sequence

$$x_1 = a, x_2 = b, \cdots, x_n = x$$

such that for $2 < i \leqq n$, each x_i bears an order relation to two preceding terms of the sequence. We show by induction on i, that x_i is expressible in the form (11).

This is obvious for $i=1, 2$. Assuming the proposition for $i<k$, where $2<k\leqq n$, we deduce it for $i=k$. We know that one of the relations

$$(x_k x_l x_m), \qquad (x_l x_k x_m) \tag{12}$$

holds for some l, $m<k$. Solving in (12) for x_k in terms of x_l, x_m we get a relation of the form

$$x_k = \lambda' x_l + \mu' x_m, \qquad \lambda' + \mu' = 1. \tag{13}$$

The hypothesis of the induction implies

$$x_l = \lambda_l a + \mu_l b, \qquad \lambda_l + \mu_l = 1, \tag{14}$$

$$x_m = \lambda_m a + \mu_m b, \qquad \lambda_m + \mu_m = 1. \tag{15}$$

Eliminating x_l, x_m between (13), (14), (15) we get an expression for x_k of the form (11). Thus the proposition is true by induction, and $x=x_n$ is expressible in the form (11).

Conversely we show that any x in the form (11) is in line ab. Suppose x expressed in form (11). Since $\mathfrak{R}'$ is linearly ordered (§6), λ is linearly related to 1, 0. Hence there exists a sequence of elements of $\mathfrak{R}'$

$$\lambda_1 = 1, \lambda_2 = 0, \cdots, \lambda_n = \lambda$$

in which each term after the second bears an order relation to two preceding terms. Let x_i be determined by

$$x_i = \lambda_i a + \mu_i b, \qquad \lambda_i + \mu_i = 1 \qquad (i = 1, \cdots, n).$$

Observe that $x_1=a$, $x_2=b$, $x_n=x$. In view of the Lemma, each term of the sequence $x_1, x_2, \cdots, x_n$ after the second bears an order relation to two preceding terms. Thus by definition, x is linearly related to a, b and belongs to line ab.

This result is quite interesting. For, considering λ as a parameter, the form (11) is a parametric equation for the "cartesian line" containing points a, b in the analytic geometry of the (unordered) field $\mathfrak{R}'$. Thus the lines determined in S by our theory of order are precisely the familiar "cartesian lines" defined by linear equations in the two dimensional analytic geometry of $\mathfrak{R}'$. Since it is well known that G3 holds for such "lines," G3 is valid in S and S is a *partially ordered geometry*.

To see whether S is a simply ordered geometry we consider the ordering of the lines of S. Relation (11) establishes a (1-1) correspondence between the set of all λ in $\mathfrak{R}'$ and the set of all x in line ab. In view of the Lemma this correspondence preserves the order relation in both directions. Thus we may say that line ab is *isomorphic* to $\mathfrak{R}'$ with respect to order. Hence all the lines of S are isomorphic as partially ordered sets and are non-simply ordered. Thus we may complete our discussion with the

THEOREM. *S is a partially but not simply ordered geometry.*

DISCUSSIONS AND NOTES

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The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

PAN-MAGIC SQUARES OF THE FOURTH ORDER

E. G. ALLEN, West Newton, Mass.

It is well known* that there are only three essentially different pan-magic (or Nasik, or diabolic) squares of the fourth order:

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

All others can be derived from these by reflections, rotations and (cyclic) translations.

It does not seem to have been noticed before that the four neighbors of any particular number are the same in all these three instances, and consequently in all the 384 pan-magic squares; *e.g.*, the number 1 is always surrounded by 8, 12, 14, 15, and the number 16 is always surrounded by 2, 3, 5, 9.

An easy method for finding these associated numbers is as follows. Form a square with the numbers from 1 to 16 in natural order, thus:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

For any given number n, locate the complementary number $17-n$. (This is always symmetrically opposite the given number.) Then take the four numbers that lie in the same row and in the same column as the complementary number but not in the same row or column as the given number. These four are the associated numbers. Thus for the given number 12 we consider the complementary number 5. The numbers in the same row as 5 but not in the same column as 12

* See e.g., M. Kraïtchik, La Mathématique des Jeux, Brussels, 1930, p. 167 (Fig. 181–183). Note that Kraïtchik made some errors in copying these squares into the English version, Mathematical Recreations, New York, 1942, p. 191.

are 6 and 7. The numbers in the same column as 5 but not in the same row as 12 are 1 and 13. Hence the numbers associated with 12 are 1, 6, 7, 13.

A given number in any selected cell, and its four proper neighbors in any one of their 4! possible permutations, determine the entire square. For, the remaining cells can be filled by observing that diagonally alternate numbers must be complementary, and that the four numbers at the corners must add to 34, like the rows, columns, and diagonals. Since the given number may be placed in any one of the 16 cells, while the four associated numbers may be permuted in 24 ways, we obtain in this manner all the $16 \cdot 24 = 384$ pan-magic squares.

CONCERNING A CERTAIN CONGRUENCE

BEN DUSHNIK, University of Michigan

1. Recently* Erdös and Niven pointed out that if the positive integers $x=1, y, z$ satisfy the congruence

$$x^3 + y^3 + z^3 \equiv 0 \pmod{xyz}, \tag{1}$$

then the integers $1, y, (y^3+1)/z$ will also satisfy (1). The derivation of this was not indicated. In this note we state and prove a theorem from which the above fact will follow as a simple corollary. Throughout we may suppose, without loss of generality, that a solution (x, y, z) of the congruence consists of positive integers which are mutually prime.

2. THEOREM 1. *If (x, y, z) is a solution of* (1), *and if u is any factor of x^3+y^3 such that*

$$u^3 \equiv z^3 \pmod{xy}, \tag{2}$$

then (x, y, u) will satisfy (1).

Proof. It is readily seen that (1) is equivalent to the system of congruences

$$x^3 + y^3 \equiv 0 \pmod{z},$$

$$x^3 + z^3 \equiv 0 \pmod{y}, \tag{3}$$

$$y^3 + z^3 \equiv 0 \pmod{x}. \tag{4}$$

From (2) combined with (3) and (4), respectively, we get

$$x^3 + u^3 \equiv 0 \pmod{y}, \tag{5}$$

$$y^3 + u^3 \equiv 0 \pmod{x}. \tag{6}$$

By hypothesis, we have

$$x^3 + y^3 \equiv 0 \pmod{u}. \tag{7}$$

Since u is clearly prime to both x and y, we get, from (5), (6), and (7),

$$x^3 + y^3 + u^3 \equiv 0 \pmod{xyu}.$$

* This MONTHLY, vol. 53, 1946, pp. 223–224.

3. To use the above theorem as a "practical" means of deriving new solutions of (1) from a given solution is not feasible, since the theorem asserts nothing about the existence of a number $u \neq z$ with the required property. There are, however, at least two cases where such a number u always exists. One is the case mentioned by Erdös and Niven; indeed, if we put

$$u = \frac{y^3 + 1}{z},$$

then u is a factor of y^3+1, and it is easy to compute that

$$u^3 \equiv z^3 \pmod{y}.$$

Another case is given by the following theorem:

THEOREM 2. *Let* $(1, y, z)$ *satisfy* (1), *and put* $u = y^2+z^2-yz$; *then* (u, y, z) *will also satisfy* (1).

Proof. By the theorem in 2, it is sufficient to show that

$$u^3 \equiv 1 \pmod{yz}, \tag{8}$$

since u is a factor of y^3+z^3. Now, $1+y^3+z^3 \equiv 0 \pmod{yz}$ implies that

$$(y^3 + z^3)^2 \equiv 1 \pmod{yz}. \tag{9}$$

Moreover,

$$(y^3 + z^3)^2 \equiv y^6 + z^6 \equiv (y^2 - yz + z^2)^3 \pmod{yz}. \tag{10}$$

Combining (9) and (10), we obtain (8).

We might add that if $1 \leqq y < z$, then $z < u$, so that (u, y, z) is a new solution.

4. Starting with the solution (1, 1, 2) of (1), we can get, by the remark of Erdös and Niven, the infinite set of different solutions

$$(1, 2, 3), (1, 3, 14), (1, 14, 915), \cdots .$$

From these last we can now get still another infinite set

$$(2, 3, 7), (3, 14, 163), (14, 915, 824611), \cdots .$$

We can thus state that (1) has infinitely many essentially different solutions (x, y, z) such that none of the numbers x, y, z is 1.

Finally, there are solutions of (1) which are not so obtainable, such as:

$$(3, 11, 679), (5, 77, 11138), (19, 26, 905),$$

and many others (obtained in a rather haphazard fashion).

ON THE FEUERBACH POINTS

R. GOORMAGHTIGH, Bruges, Belgium

1. The construction given by S. M. Karmelkar, this MONTHLY, vol. 53, 1946, p. 206, for the in-Feuerbach point Z of a triangle is another form of Mannheim's well known theorem (Bulletin des Sciences mathématiques, 1902, p. 95).

Let, as in the recalled Note, D be the mid-point of BC, P the foot of the altitude from A, I the in-center, L its projection on BC, X the point where the perpendicular bisector of BC meets PI, and Y the point where DI cuts AP.

According to Mannheim's theorem, if T is the image of L in I, the join of T to the mid-point A' of AI passes through Z. But as $AY=IL$, TA' passes through Y; further $TI=IL$, and XD is perpendicular to BC, so this proves that the line XY in the considered construction coincides with the Mannheim line TA'.

2. The fact that the point Y obtained by taking AY equipollent to IL belongs to the Mannheim line corresponding to BC may be extended to the case of the orthopole of any circumdiameter d of ABC, according to the following generalization of Mannheim's theorem which I have given in the Journal de Mathématiques élémentaires (Vuibert), 1913–1914, p. 69:

If Q' is the counter-point of any point Q on d and if the pedal circle $Q_a'Q_b'Q_c'$ of Q' meets again $Q_a'Q'$ at S, then the straight line joining S to the mid-point A'' of AQ' passes through the orthopole of d.

It follows that $A''S$ cuts the altitude AP at a point Y' such that $AY'=SQ'=QQ_a$, Q_a being the projection of Q on BC.

When d is the circumdiameter passing through I, the orthopole of d is the in-Feuerbach point.

3. If Q and Q' are such that QQ' is parallel to BC, these points lie on a cubic Γ, anallagmatic in triangular inversion and having as normal equation

$$cy(x^2 - z^2) + bz(x^2 - y^2) = 0.$$

In that case $Q'S=Q_a'Q'$ and we find the following generalization of Karmelkar's construction:

Let Q be a point on the cubic Γ, Q' its counter-point, Q_a and Q_a' the projections of Q and Q' on BC, Y' the point on the altitude AP such that $AY'=QQ_a$, D' the intersection of $Y'Q'$ with BC, X' the point where the perpendicular at D' to BC meets PQ'; then the projection Z' of Q_a on $X'Y'$ is the orthopole of the circumdiameter passing through Q.

4. This proves that the considered construction applies also to the outer-Feuerbach points.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

A Trigonometric Approximation

A paper on the calculation of the trigonometric ratios printed privately (Elm Press, Calcutta) by Jatis Chandra Ray, a practising neurologist of Calcutta, has been brought to this editor's attention. Although expressed in somewhat different notation, his formulas for calculating the trigonometric ratios for fractions of a right angle are equivalent to the approximation

$$\tan\frac{\pi}{2}x \sim \frac{x}{1-x}\left(\frac{5-3x}{2+3x}\right)^{1/2} \tag{1}$$

and the corresponding formulas immediately derivable for $\sin x$ and $\cos x$. The approximations are exact for angles of 0°, 30°, 45°, 60°, 90°—that is, for $x=0$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, 1—and are not changed by replacing an angle by its complement. The slope in (1) for $x=0$ is $\frac{1}{2}\sqrt{10}$ instead of $\frac{1}{2}\pi$, and for $x=\frac{1}{2}$ is $22/7$ instead of π. For angles between 0° and 45° the maximum error is about 0.00048 (near sin 7.5°).

Ray gives an improved approximation by multiplying the exponent $\frac{1}{2}$ in (1) by the factor $1+(3-2x)(2-3x)(3x-1)/420$, but states erroneously that this is "mathematically accurate." The error may be much smaller, but no such finite expression for $\tan(\pi x/2)$ could be exact. With this exponent, the slope at the origin is reduced to 1.569 instead of $\pi/2$.

It may be of interest to note that the approximation

$$\sin\frac{\pi}{2}x \sim x\,\frac{11-3x^2}{7+x^2} \tag{2}$$

gives a better fit for angles less than 30° than Ray's first approximation, and does not involve the extraction of a square root.

CLUB REPORTS 1945–46

Pi Mu Epsilon, Hunter College

Under the directorship of Dr. Annita Tuller, papers were presented on important aspects of geometry.

Geometric points of view, including: *Transformation group point of view, Metric, Affine, Projective geometry, Circular transformations, Linear fractional transformations of a complex variable, Introduction to the study of geometry from the metric point of view.*

Geometric fields of study, including: *Topology*, *Differential geometry of plane curves*, *Space curves*, and *Surfaces*, *Algebraic geometry*.

The following undergraduates contributed to the discussions: Chang Jung, Rae Adelson, Mary Rosania, Edith Weiss, Theresa Danielson, Florence Jaffe, Lorraine Kruglov, Rita Gross, Mary Greene, Carol Podell, Anne Stern, Leila Rubashkin, Josephine Rodstrom, and Ruth Weihrauch.

At initiation meetings in Hunter's North Lounge, 22 members were initiated during the year. The topic at the first of these was

The isograph, by Mr. Robert Dietzold of Bell Laboratories. The second initiation on March 31 was a part of the first meeting of the Metropolitan Intercollegiate Convention, sponsored by the chapters of Hunter College (*Beta*), Brooklyn College (*Gamma*), and New York University (*Delta*), and attended by approximately 200 faculty and students from colleges throughout the city. Papers presented were

Topics in Mathematics, by Professor Edward Kasner of Columbia

Applications of the vector operator del, by Professor Carel van der Merwe of New York University.

Lorraine Kruglov received a copy of Klein's *Elementary Mathematics from an Advanced Standpoint* for presenting the best paper of the year.

A theater party was organized between semesters.

Officers for 1945–46 were: President, Mary Greene; Corresponding Secretary, Sally Rothstein; Recording Secretary, Carol Podell; Treasurer, Leila Rubashkin. Newly elected officers for 1946–47 are: President, Cecile Salwen; Corresponding Secretary, Annette Drucker; Recording Secretary, Ruth Friedman; Treasurer, Mildred Vogt. Professor Lucille Anderson will be Director.

Mathematics Club, Hunter College

An active program included a talk by Professor Edward Kasner, and three other faculty talks

The role of curves in the history of mathematics, by Professor Carl B. Boyer of Brooklyn College

Finite geometry, Professor Jewell Hughes Bushey

The relation of mathematics to the sciences, by Professor Bushey. Reports were given by students on the topics

Transcendentals and imaginaries

Chinese mathematics

Paradoxes

Fun with eight numbers

Polar coordinates

Hyperbolic functions

Vector analysis

Social activities consisted of a Christmas party and a Spring party.

Officers were: President, Sally Rothstein; Vice-President, Dorothy Beck; Treasurer, Cecile Cohen; Secretary, Wilhelmina Fluhr; Publicity Chairman, Ina Pinchuk. The Faculty Adviser was Miss Isabel McLaughlin.

Pi Mu Epsilon, Louisiana State University

The schedule of events included an organization meeting, two initiation meetings, a picnic, and the following program meetings:

Radar, by Dr. L. W. Morris
Projectiles in war, by Dr. Paul White
Prominent mathematicians, by Newton Hawley
Lengths and areas, by Dr. H. L. Smith
Teaching G.I.'s in England, Dr. N. E. Rutt
Empirical probability, by Dr. Peter Carmichael
The problem of the infinity in the eighteenth century, by Dr. Eric Voegelin
Inversion, by Zoe Ellen Sanders.

Winners of the two $15 *Pi Mu Epsilon* Awards were: Newton Hawley, the senior having the best mathematical record from the standpoint of quality of work done and quantity of work taken; and Mary Ellen Rickey, the freshman making the highest score in a competitive examination. The examination committee consisted of Elizabeth Cathey, chairman, Miss Yvonne Jones, Charles McCleskey, Katherine Formusa.

Several new books were added to the *Pi Mu Epsilon* collection of histories, biographies, and popular titles given to the Mathematics Library.

Officers and committee chairmen for 1945–46 were: President and Membership Chairman, Lollie Belle Bienvenu; Vice-President and Program Chairman, Newton Hawley; Secretary, Katherine Formusa; Treasurer and Entertainment Chairman, Lorraine Schertz. Officers for 1946–47 are: President, Ben E. Mitchell; Vice-President, Charles McCleskey; Secretary, Beverly Jean Russell; Treasurer, Kathryn Jumonville. Dr. Houston Karnes served both years as Corresponding Secretary.

Kappa Mu Epsilon, College of St. Francis

The Illinois Delta Chapter of *Kappa Mu Epsilon* was installed at the College of St. Francis, Joliet, on May 22, 1945 by Professor Urbancek of the Illinois Gamma Chapter. Meetings were held on the second Thursday of every month. Topics chosen for discussion were

Zeno's paradoxes
Modern geometry
The discriminant of the quadratic, cubic, and quartic
Applications of the differential calculus
The disciplinary, utilitarian, and cultural aims of the teaching of mathematics

Officers for 1945–46 were President, Lillian Koss; Vice-President, Helena Weigand; Secretary, Margaret Markiewicz; Treasurer, Jeanne Erlenborn; Corresponding Secretary and Faculty Sponsor, Sister M. Claudia Zeller. Six new members were initiated and the following new officers were installed on May 9, 1946: President, Betty Fiene, Vice-President, Marian Masters; Secretary, Betty Lanoue; Treasurer, Patricia Young.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Sequential Analysis of Statistical Data: Applications. Prepared by the Statistical Research Group, Columbia University. Columbia University Press, 1945. $6.25.

Sequential analysis is an important new statistical technique developed initially by A. Wald in 1943 for use in war research. Its basis may be illustrated for non-statisticians by considering the first application described in this book.

In the process known as acceptance sampling, a random sample from a batch of manufactured articles is inspected in order to determine whether the batch is acceptable. Let us suppose that each article in the sample, after inspection or testing, may be classified as "good" or "bad." In an early type of sampling plan, with which sequential sampling will be contrasted, N articles are inspected. The batch is accepted if b or less of these articles are bad and is rejected otherwise. How effective is this sampling plan? This may be seen from a simple formula. If p is the unknown proportion of bad articles in the batch and if the batch is many times larger than N, the probability that the batch is accepted will be given, to a close approximation, by the binomial summation:

$$\sum_{x=0}^{b} \frac{N!}{x!(N-x)!} p^x(1-p)^{N-x}.$$

As p increases from 0 to 1, the probability of acceptance, as given by this formula, decreases steadily from 1 to 0. Nevertheless, and this is true of any sampling plan, some batches with very low p will be rejected and some with relatively high p will be accepted. By choice of N and b, however, a partial control can be obtained on these two types of error. In practice, it is customary to fix two points on the curve. That is, N and b are calculated so that the probability of rejecting a batch with proportion p_1 does not exceed α, while the probability of accepting a batch with proportion p_2 does not exceed β. The quantities p_1, p_2, α, and β are of course selected by the inspector. Since p_1 usually corresponds to a proportion which it is highly desirable to accept and p_2 to one which it is highly desirable to reject, α and β are usually made small.

For many manufacturing processes involved in war production, inspection was both costly and destructive. It was therefore exigent to reduce the inspection to a minimum. In this respect the plan described above is somewhat wasteful, as had been realized previously. For example, with a poor batch, more than b bad articles might appear long before the total N had been inspected. The

sequential method was developed in response to this need for minimum inspection.

With the sequential method, two fundamental changes appear. First, instead of waiting for a decision until all N articles have been inspected, the results to date are examined as each successive article is inspected. From the examination one of three decisions is made: (i) to accept the batch, (ii) to reject the batch, (iii) to continue further inspection. The sampling terminates when a decision for acceptance or rejection is reached. The second important change, which is a corollary of the first, is that the total size of sample N is no longer fixed, but is a random variable depending on the quality of the batch. In general, if p is very low or very high, a decision will be reached quickly. With a borderline p larger samples are needed.

The mathematical problem to be solved in constructing a sequential method is that of developing the rules for decisions (i), (ii) and (iii) above. These rules must satisfy the two "control" conditions mentioned previously, namely that the probability of rejecting a batch with proportion p_1 must not exceed α, and the probability of accepting a batch with proportion p_2 must not exceed β. Further, for the present purpose, the rules must be such that the average amount of inspection needed is a minimum. The rules found by Wald to satisfy these conditions are surprisingly simple. If at any state n articles have been inspected and d are bad, the batch is accepted if

$$\frac{p_1^d(1-p_1)^{n-d}}{p_2^d(1-p_2)^{n-d}} \geqq \frac{1-\alpha}{\beta}$$

and rejected if this quantity is less than or equal to $\alpha/(1-\beta)$. If the quantity falls between these limits, sampling is continued. On taking logarithms the rules become still simpler for practical application, since it is clear that they may be reduced to the following: Accept if $d \leqq -h_1+sn$; Reject if $d \geqq h_2+sn$, where h_1, h_2, and s are simple functions of p_1, p_2, α and β. Consequently the inspector may either apply to a table showing the decisive values of d for given n, or plot the points (d, n) on a graph containing the two decisive parallel lines.

Sequential analysis may be applied to a wide variety of problems in which one of two courses of action is chosen on the evidence provided by a sample of data. Other applications described in this volume are to the choice between two batches or methods according to the proportions of bad articles which they produce, to the acceptance or rejection of batches where quality is measured on a continuous scale, and to the question whether the variability within a batch, as measured by the standard deviation, meets acceptable standards. In each case the principal object is to reduce the average amount of data necessary for a decision with controlled risks of error. While little definite information is given about the actual size of this reduction, values of 40 to 60 percent appear to be attained in certain cases as compared with sampling where N is fixed in advance.

The present volume comprises seven separate pamphlets, largely self-contained. Apart from the first, which is introductory, and the last, which gives certain formulae common to all applications, each pamphlet deals with a specific application. The publication is evidently directed at potential users of the methods who are expert neither in mathematics nor statistics. The rules for using the method are stripped of all mathematical complexities and supplementary tables are given wherever helpful. No attempt is made to present details of the theory, though the common-sense justification for the procedures is included. The mathematician who wishes to learn about sequential analysis will probably find it more profitable to begin with Wald's paper "Sequential tests of statistical hypotheses," Annals of Mathematical Statistics, Vol. 16, part 2, 1945. Thereafter, he may be interested in perusing this volume in order to see how the authors have tackled the many problems involved in putting the technique into the hands of the non-mathematician.

W. G. COCHRAN

Lectures on Probability and Statistics. By E. L. Dodd. Austin, Texas, The University of Texas Press, 1945, 44 pages. No charge.

This includes first a short biographical sketch of Professor Dodd, with a bibliography. Then there are three of his papers, the first two being lectures which were found in manuscript form among his effects, and the third being a copy of a paper sent to the Seventh American Scientific Congress, Mexico City, in 1935. The two lectures are elementary in nature and describe with great clarity various kinds of means and their properties. The last paper, entitled "A survey of statistical means or averages," is the longer and more important part of this brochure. Needless to say, the survey is authoritative, coming from one who had spent much of his mathematical life studying and writing about means. "In general analysis a mean has always signified a number not less than the least number (greatest lower bound) of a set and not greater than the greatest (least upper bound). As regards means used for statistical purposes, however, the notion of intermediacy was originally not fundamental—in all probability. Means arose, more likely, in connection with certain problems requiring a substitution of a single number for a set of numbers. O. Chisini and others have stressed the notion of a substitution, setting forth the equation

$$F(M, M, \cdots, M) = F(x_1, x_2, \cdots, x_n)$$

to define the mean M of $x_1, x_2, \cdots x_n$." For example, if we want to define the arithmetical mean we let F be the sum of the numbers in the parenthesis following it. That is, we can say that the arithmetical mean is one number substituted for each of n numbers, and is such that the sum of these n numbers is not changed by the substitution. Likewise, if we want to define the geometric mean we let F indicate the product of the numbers following it, and then the product is not changed by the substitution. "When M is expressed explicitly as

$$M = f(x_1, x_2, \cdots, x_n),$$

it turns out that the only restriction placed upon f is

$$f(x_1, x_2, \cdots, x_n) = x.$$

Thus, a *statistical* mean need not be intermediate." Many useful means result from this definition. Many means involve summation and the summation may be generalized into an integration. The survey paper also contains a section which expresses a general definition of a statistical mean in the language of point sets.

B. H. CAMP

Elementary Applied Aerodynamics. By P. E. Hemke. New York, Prentice-Hall, Inc., 1946. 7+231. pages. $3.25.

The author succeeds very well in carrying out the objectives announced in his preface: "*Elementary Applied Aerodynamics* has been written to meet the requirements of a first course in technical applied aerodynamics. Such a course is usually given to students who have had preliminary training in mathematics, physics, and mechanics. An adequate preparation in these subjects eliminates the need for an introductory course that may have been simplified to such an extent that it loses its effectiveness and fails to challenge the student. . . . The material is presented in comparatively simple fashion, but this treatment is not merely descriptive when a fundamentally sound analytical treatment is desirable. Illustrative examples are freely used and fully worked out. . . . Numerous problems are included at the end of each chapter."

The table of contents is as follows: 1. Units and Dimensions; 2. Physical Properties of Air; 3. Flow of an Ideal Fluid; 4. Lift and Moment of an Airfoil in an Ideal, Two-Dimensional Flow; 5. Three-Dimensional Flow Around an Airfoil, Induced Effects; 6. Viscosity Effects; 7. Airfoils and Their Properties; 8. Compressibility Effect on Airfoil Characteristics; 9. Propellers; 10. Performance of a Conventional Airplane; 11. Helicopters. Chapter 1 and 2 give the usual discussion of dimensional analysis, pressure and density variation with altitude, viscosity of air, and compressibility of air. They are noteworthy for containing numerical values and units for the standard constants and conversion factors needed to solve problems later in the text; in most texts the values are scattered throughout the text and difficult to locate or else are not given and must be derived before solving the problem.

Chapters 3 and 4 develop the two-dimensional ideal flow for airfoils with particular emphasis on the stream function. The diagrams showing chordwise distributions are helpful. The introduction of the idea of drag is somewhat confused while the discussion of moment and aerodynamic center seems longer and more involved than necessary. Mathematically, the substitution to integrate the integral at the bottom of page 44 needs justification since it involves division by zero at the lower limit.

The discussion of three-dimensional flow, induced effects and viscosity effects in chapters 5 and 6 are above average. The laminar and turbulent boundary layers are examined in detail. Chapter 7 describes some of the characteristics of standard NACA airfoils and gives a good treatment of the flap effects on airfoil characteristics. However, no mention is made of the effects of slots, spoilers, dive brakes, boundary layer control devices, and sweep back on airfoil characteristics. Chapter 8 gives an excellent introduction for the beginner to the subject of compressibility effects on airfoils. Naturally, only the essential outline of the problem of compressibility could be given, for this problem is and will continue to be for a long time the subject of intense study and research by the wind tunnels and aerodynamists of the world.

Chapter 9 outlines the basic procedures in determining propeller characteristics and points out the essential differences between the performance of fixed-pitch and constant-speed propellers. Chapter 10 considers the simplified conventional performance calculations of a conventional airplane, covering take-off time and distance, rate of climb, minimum and maximum flight speeds, range and endurance, landing speed and landing run. Chapter 11 sketches the performance problems of helicopters, discussing hovering and forward flight, and indicating the effects of blade flapping.

Although the book is not difficult to read, there are a considerable number of typographical errors, particularly subscripts being printed level with the symbol. Some of these would prove confusing to the beginner. Also, one misses the smooth diction and well connected explanations of the best texts and is reminded of a lecturer's notes. Nevertheless, the book will undoubtedly take its place as one of the best at the present time for the beginner in aerodynamics.

B. E. GATEWOOD

NEW BOOKS RECEIVED

Calculus. Second Edition. By F. H. Miller. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1946. 16+416 pages. $3.50.

Courbes Algébriques sur Corps Germés de Caractéristique Quelconque. (Acta Salmanticensia, Serie de Ciencias, Seccion de Matematicas, No. 1). By G. Anocchea, Madrid, Univ. de Salamanca, 1946. 39 pages. 10 ptas.

The Development of Trigonometry from Regiomontanus to Pitiscus. (Dissertation, University of Michigan, 1944). By Sister Mary Claudia Zeller. Joliet, Ill., Sisters of St. Francis of Mary Immaculate, 1946. 6+119 pages.

Die Mathematische Denkweise. By A Speiser. Basel Birkhäuser, 1945. 132 pages. s. fr. 14.50.

Les Espaces Normaux. By M. Hachtroudi. Université de Téhéran, 1945. 83 pages.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 736. *Proposed by Paul Erdös, Stanford University*

Let $a_1<a_2<\cdots<a_k\leqq n$, where $k>[(n+1)/2]$, be k positive integers. Then $a_i+a_j=a_r$ is solvable. (*Cf.* No. 3739 [1937, 120]).

E 737. *Proposed by V. L. Klee, Jr., University of Virginia*

Establish the divergence of the series $\sum_{n=1}^{\infty} n^{-a}\cos(b\log n)$ for all values of $a\leqq 1$, regardless of the value of b.

E 738. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the four spheres passing through the Monge point and the nine point circles of the faces of a tetrahedron are equal to each other.

E 739. *Proposed by L. M. Kelly, University of Missouri*

An ellipse inscribed in the triangle ABC is tangent to AB at D. Show that the midpoints of CD and AB are collinear with the center of the ellipse.

E 740. *Proposed by Esther Szekeres, Shanghai, China*

Let there be given five points in the plane. Prove that we can select four of them which determine a convex quadrilateral.

SOLUTION

Overlapping Progressions

E 696 [1945, 578]. *Proposed by C. H. Wolfe, Lakeside High School, Ohio*

Find four positive integers a, b, c, d such that a, b, c are in geometric progression, b, c, d in arithmetic progression, and $c+d=44$.

Solution by Maud Willey, Denton, Texas. Solutions must satisfy the equations (1) $ac=b^2$, (2) $b+d=2c$, (3) $c+d=44$. Eliminating b and d we obtain $ac=9c^2-264c+1936$. Hence c divides $1936=2^4 11^2$. But from (3), $c<44$, and from (2) and (3), $c>14$. The possible values of c are thus 16 and 22, giving the solutions 1, 4, 16, 28 and 22, 22, 22, 22.

If the restriction that the integers be positive be lifted, there are thirty solutions in all, corresponding to the thirty factors of 1936. In particular, $c=44$

leads to the only other solution in non-negative integers, namely 176, 88, 44, 0.

Also solved by F. A. Alfieri, D. W. Alling, Murray Barbour, P. T. Bateman, R. C. Buck, W. E. Buker, H. N. Carleton, P. L. Chessin, R. E. Crane, J. H. Cross, Monte Dernham, William Douglas, S. E. Field, Mary Goldsmith, Sidney Glusman, Herrick Greenleaf, N. G. Gunderson, Francis Hall, R. H. Hoskins, H. K. Humphrey, Aida Kalish, M. Kirk, H. L. Lee, H. R. Leifer, Eric Michalup, W. Nicholson, C. C. Oursler, S. T. Parker, J. V. Pennington, P. A. Piza, P. W. A. Raine, W. C. Rufus, F. W. Saunders, E. D. Schell, Nathan Schwid, M. J. Sheehy, E. P. Starke, P. D. Thomas, J. A. Tierney, F. Underwood, R. H. Urbano, Hazel Wilson, and the proposer. Monte Dernham suggested the more difficult associated problem: *Replace* 44 *by n. For what values of n are there exactly m solutions?*

Turning an Automobile

E 701 [1946, 36]. *Proposed by N. S. Mendelsohn, Queen's University*

Suppose the distance between the front and rear wheels of a car is b, and the distance between wheels from side to side is a. The right front wheel is turned through a counterclockwise angle θ from its neutral position. Through what angle must the left front wheel be turned so that in the car's subsequent motion all four wheels will roll without slipping? Describe the motion.

I. *Solution by Herbert Reisman, Illinois Institute of Technology.* If the rear axle and the axle of the right front wheel are extended, they will meet at a point P, the instantaneous center of rotation of the car. To prevent slipping, the axle of the left front wheel must also pass through P. Denoting the counterclockwise angle of the left front wheel, measured from its neutral position, by ϕ, it is easily shown that the angle between the rear axle and the left front wheel axle is also equal to ϕ. Denoting the distance from the left rear wheel to the point P by x, the following relationships are obtained:

$$\tan \phi = b/x, \qquad \tan \theta = b/(a + x).$$

Eliminating x and solving for ϕ, we find

$$\phi = \arctan [b/(b \cot \theta - a)].$$

The car will describe a circle about point P.

II. *Solution by Frank Hawthorne, Columbia University.* Since there is to be no slipping, the direction of motion of the center of each wheel must be in the plane of the wheel. If $\theta \neq 0$, the perpendiculars to the planes of the wheels at their respective centers must be concurrent in a point P which is the center of rotation for the car as a whole. P lies on the line of the rear axle, for the rear wheels are normally fixed (except for rotation) with respect to the automobile. In the following we shall consider values of θ between 0° and 90°, although such a range is more than is normally provided.

Case 1. $0 < \theta < \arctan b/a$. P is to the left of the left rear wheel. The corre-

sponding angular displacement of the left front wheel is $\phi = \arctan [b/(b \cot \theta - a)]$. The rotation of the car about P is such that the wheels all rotate in the same sense.

Case 2. $\theta = \arctan b/a$. The car rotates about its left rear wheel, and $\phi = 90°$.

Case 3. $\arctan b/a < \theta < \arctan 2b/a$. P lies on the rear axle between the left wheel and the center. The two front wheels and the right rear wheel rotate in the same sense. The left rear wheel rotates in the opposite sense. For this case $\phi = \arctan [(a - b \cot \theta)/b] + 90°$.

Case 4. $\theta = \arctan 2b/a$. P is at the midpoint of the rear axle. No motion will occur in a rear wheel drive car. For a front wheel drive the car would rotate about this point, the wheels turning oppositely.

Case 5. $\arctan 2b/a < \theta < 90°$. P is on the rear axle to the right of the center. The two front wheels and the right rear wheel rotate in the same sense ("reversed" for "ahead" driving) while the left rear wheel rotates in the opposite sense. Here we have $\phi = \arctan [(a - b \cot \theta)/b] + 90°$.

Case 6. $\theta = 90°$. The car rotates about the right rear wheel and $\phi = \arctan a/b + 90°$.

Also solved by E. F. Allen, W. G. Brady, J. H. Cross, Vladimir Karapetoff, W. A. Rees, P. D. Thomas, and the proposer.

The proposer stated that around the year 1920 numerous papers connected with the turning of automobiles appeared, and, at that time, a number of mechanical devices were invented which would automatically cause the front wheels to align themselves properly when turning. The problem was not appreciated in the very earliest days of the automobile, and was not considered in the days before the automobile, as wagons were constructed with their front wheels perpendicular to the axle, which was pivoted at its midpoint.

Karapetoff remarked that, since the two rear wheels follow circles of different radii, peripheral slippage can be prevented only if the rear axle is broken in two by a differential, as is the case with all trucks and automobiles, or if the two rear wheels are mounted on individual center pins, and have no common revolving axle, as in the case of horse-driven vehicles.

Consecutive Cubes with Difference a Square

E 702 [1946, 36]. *Proposed by J. Brenner, Palmer Laboratory, Princeton, N. J.*

The smallest consecutive cubes whose difference is a square are 8^3 and 7^3. What is the next pair?

Solution by E. P. Starke, Rutgers University. By hypothesis we have

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1 = r^2, \tag{1}$$

which may be written

$$(2r)^2 - 3(2n+1)^2 = 1. \tag{2}$$

Since $(2^2 - 3 \cdot 1^2)^k = 1$, r_k and n_k will satisfy (2) if

$$2r_k - \sqrt{3}\,(2n_k + 1) = (2 - \sqrt{3})^k,$$
$$2r_k + \sqrt{3}\,(2n_k + 1) = (2 + \sqrt{3})^k,$$

whence

$$4r_k = (2 + \sqrt{3})^k + (2 - \sqrt{3})^k, \tag{3}$$
$$2\sqrt{3}\,(2n_k + 1) = (2 + \sqrt{3})^k - (2 - \sqrt{3})^k,$$

where k is an odd integer if r_k is to be integral. From (3) it is not difficult to obtain

$$n_{k+4} = 14n_{k+2} - n_k + 6,$$

and thence, with $n_1 = 0$, $n_3 = 7$, we compute $n_5 = 104$, $n_7 = 1455$, $n_9 = 20272$, *etc.* It can also be shown without difficulty that this procedure gives all solutions of (1).

It is worth noting that (1) is a relation for the triangle with integral sides n, $n+1$, r in which the angle opposite r is 120°.

Also solved by D. W. Alling, Norman Anning, Murray Barbour, P. T. Bateman, R. C. Buck, W. E. Buker, Monte Dernham, William Douglas, N. G. Gunderson, R. Hamming, H. L. Lee, W. Nicholson, S. T. Parker, Walter Penney, J. V. Pennington, P. A. Piza, W. C. Rufus, Theodore Running, E. D. Schell, Elijah Swift, F. Underwood, R. H. Urbano, B. F. Yanney, and the proposer.

Bateman, Schell, Alan Wayne, and Yanney pointed out that this problem had been solved by C. Richaud, and gave as reference Vol. II, p. 580, of Dickson's *History of the Theory of Numbers.*

Running noted that for the first few solutions r takes on successively the values 13, 181, 2521, 35113, 489061, 6811741. In each of these, r has the form $b^2 + (b-1)^2$. Does r assume this form for every solution in positive whole numbers?

Anning stated that the values of $n+1$ satisfy the recurrence relation

$$u_1 = 1, \qquad u_2 = 8, \qquad u_3 = 105,$$
$$u_n = 15u_{n-1} - 15u_{n-2} + u_{n-3}, \qquad n = 4, 5, 6, \cdots.$$

Most solvers applied the standard theory of the Pell equation to equation (2) above.

A Square of the Form abbc

E 703 [1946, 36]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In which scales of notation can a four-digit number *abbc* be the square of a two-digit number *mn*, if $c = a + 1$ and $n = m + 1$? (*Cf.* E 652 [1945, 42 and 459]).

Solution by E. P. Starke, Rutgers University. The method of the former problem can be used here. Let r be the radix, so that by hypothesis we have

$$ar^3 + br^2 + br + c = (mr + n)^2, \tag{1}$$

which, with $c=a+1$ and $n=m+1$ can be easily reduced to

$$a(r - 2) + b - m^2 = (2m + b - 3a)/(r + 1).$$

Since the right member is an integer, say I, this gives

$$a(r - 2) + b - m^2 = I, \tag{2}$$

$$2m + b - 3a = I(r + 1), \tag{3}$$

and, upon elimination of b,

$$(a + I)(r + 1) = m^2 + 2m + I. \tag{4}$$

Evidently $|I| \leqq 2$. Suppose $I<0$. From (3) $a>m$, which contradicts (2) because of $m<n<r$. Suppose $I=2$. Upon solving (4) for r, substituting in (2) and putting $b<r$, we obtain

$$m^2 - 2(a + 1)m + (3a^2 + 7a + 4) < 0,$$

which is impossible. Hence $I=0$ or 1. From (4) and the hypothesis we note that

$$1 \leqq a < m \leqq r - 2. \tag{5}$$

Let $I=1$. Then (4) becomes $(a+1)(r+1)=(m+1)^2$, and if $(m+1)^2=u^2v^2k^2$, we can put

$$a = u^2k - 1, \qquad r = v^2k - 1, \qquad m = uvk - 1,$$

and, from (3),

$$b = 3u^2k - 2uvk + v^2k - 1,$$

where $a>0$ and $b<r$ require $uk>1$, $2v>3u$. Thus r can be any number (except 3 and 8) such that $r+1$ is divisible by a square greater than unity.

Let $I=0$, then from (3) and (4) we have

$$b = 3a - 2m, \qquad a(r + 1) = m^2 + 2m. \tag{6}$$

If now $b=0$, we have $3a=2m$, $r-2=3m/2$, whence m is a multiple of 6 and we can put

$$m = 6k, \qquad a = 4k, \qquad r = 9k + 2.$$

If $b=4$, we have $3a=2m+4$, $r+1=3m/2$, whence $m+2$ is a multiple of 6 and we can put

$$m = 6k + 4, \qquad a = 4k + 4, \qquad r = 9k + 5.$$

Other values of b are possible but for each there is at most a finite number of values of r, and no general formula appears. However, from (6) it is easy to obtain solutions in terms of a single parameter by setting $a=2b$, $a=3b$, $2a=3b$, *etc.* The following are a few of them: $(a,\ b,\ m,\ r)=(9k+3,\ 3k+1,\ 12k+4,$

$16k+7)$, $(16k+8, 8k+4, 20k+10, 25k+14)$, $(9k+3, 6k+2, 21k+7, 49k+20)$, $(49k+28, 7k+4, 70k+40, 100k+59)$.

Also solved by W. E. Buker, H. L. Lee and the proposer.

The Sum of a Series

E 704 [1946, 36]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Sum the series $1+1/3-1/5-1/7+1/9+\cdots$.

I. *Solution by P. T. Bateman, Philadelphia, Pa.* By setting $x=\pi/4$ in the well known Fourier series

$$\sin x + (\sin 3x)/3 + (\sin 5x)/5 + \cdots = \pi/4, \quad (0 < x < \pi)$$

we find that

$$1 + 1/3 - 1/5 - 1/7 + 1/9 + \cdots = \pi\sqrt{2}/4.$$

II. *Solution by E. P. Starke, Rutgers University.* The geometric series

$$(1 + x^2) - (x^4 + x^6) + (x^8 + x^{10}) - \cdots = (1 + x^2)/(1 + x^4),$$

is convergent, and has the indicated sum, for $|x| < 1$. Therefore, by integration, the series

$$x + x^3/3 - x^5/5 - x^7/7 + x^9/9 + \cdots \tag{1}$$

converges for $|x| < 1$ to

$$\int_0^x [(1 + x^2)/(1 + x^4)]dx = \tfrac{1}{2}\sqrt{2} \arctan [\sqrt{2}x/(1 - x^2)].$$

Since series (1) also converges for $x=1$, we have, by Abel's continuity theorem,

$$1 + 1/3 - 1/5 = 1/7 + 1/9 + \cdots = \lim_{x\to 1} \tfrac{1}{2}\sqrt{2} \arctan [\sqrt{2}x/(1 - x^2)]$$
$$= \pi\sqrt{2}/4.$$

III. *Solution by C. D. Olds, San Jose State College.* From the well known expansion (*cf.* Titchmarsh, *Theory of Functions*, p. 113)

$$\csc z = \frac{1}{z} + 2z\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2\pi^2 - z^2},$$

we immediately obtain

$$\frac{\pi}{n} \csc \frac{m\pi}{n} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \cdots.$$

Putting $n=4$ and $m=1$, we get

$$\pi\sqrt{2}/4 = 1 + 1/3 - 1/5 - 1/7 + 1/9 + \cdots.$$

Also solved by D. W. Alling, W. G. Brady, R. C. Buck, W. E. Byrne, W. J. Combellack, H. E. Fettis, Harley Flanders, R. Hamming, H. A. Luther,

Solomon Mitchell, W. O. Pennell, Elijah Swift, P. D. Thomas, F. Underwood, and the proposer.

Olds and the proposer furnished, as an early reference, Newton's second letter to Oldenburg, 1676. The proposer also located the series as an exercise in Chrystal's *Algebra*, part 2, p. 335. W. G. Brady pointed out that the series also appears as problem 32, p. 299, in Hobson's *Plane Trigonometry*, and as problem 108c, p. 268, in Knopp's *Theory and Application of Infinite Series* (1928). Hobson gives the correct answer while Knopp gives $\sqrt{2\pi}/4$. Olds further mentioned that the series is closely related to Gregory's series for $\pi/4$, and belongs to a class of such series involving the reciprocals of the natural numbers. These were extensively studied by J. W. L. Glaisher who obtained many such series (*cf.* Messenger of Math., vol. 32, pp. 12–30). Pennell presented a number of allied series obtained from Fourier expansions. Bateman called attention to a general method given in article 107 of Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*, and to some results in Landau's *Vorlesungen über Zahlentheorie*, Band I, Vierter Teil, Kapitel 8, S. 174–178.

Construction of a Quadrangle

E 707 [1946, 97]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct a quadrangle having given the lengths of the sides and of the line joining the midpoints of the diagonals.

Solution by H. E. Fettis, Dayton, Ohio. The problem as here interpreted could more accurately be stated: To construct a complete quadrangle given the lengths of four of the sides and of the line joining the midpoints of the remaining two.

Let AB be the segment joining the midpoints of two sides. The construction may then be effected as follows:

Choose any two of the four given sides as opposite, and construct the parallelogram $AMBM'$ on AB as diagonal, having its non-parallel sides equal respectively to half of each of the two sides of the quadrangle chosen as opposite. In a similar manner, construct the parallelogram $ANBN'$, using half the remaining sides of the quadrangle as non-parallel sides of the parallelogram. Through M and N draw lines respectively parallel to AN and AM. Through M' and N' draw lines respectively parallel to BN and BM, which will intersect the two previously drawn parallels to form the required quadrangle.

The above construction may be effected in either of *two* ways, since the second parallelogram may assume two distinct positions for a fixed position of the first, but changing the position of the first parallelogram evidently will not yield a solution distinct from the first two. Thus, of the four possible figures that could result from the above constructions, only two are distinct.

In addition, there are two other possible choices of pairs of opposite sides. The total number of solutions therefore cannot exceed six.

It is clear, however, that six actual solutions are possible only if triangles can be constructed with AB and half of each of the two shortest sides, and with

AB and half of the shortest and longest sides. If either of the above triangles, but not both, is a flat triangle, the quadrangle has two parallel sides.

Also solved by G. W. Courter, J. H. White, and the proposer. An algebraic approach was indicated by H. N. Carleton.

The solutions by Courter and White were similar to the one given by Fettis. The proposer utilized the Miquel point of a complete quadrilateral as follows. Let the opposite sides BC and DA, AB and CD of a quadrangle $ABCD$ meet, respectively, in E and F, and let P be the Miquel point of the complete quadrilateral thus formed. If I and J are the midpoints of AC and BD we can obtain the relations

$$PA = AB \cdot AD/2IJ,\quad PB = BC \cdot AB/2IJ,\quad PC = BC \cdot CD/2IJ,\quad PD = AD \cdot DC/2IJ$$

The first two of these permit us to draw triangle ABP. The last two then give us C and D.

This problem occurs as Ex. 300 in Petersen's *Methods and Theories for the Solution of Problems of Geometrical Constructions.*

A Number Prime for Several Bases

E 708 [1946, 97]. *Proposed by M. P. de Regt, Todd Pacific Shipyards, Tacoma, Washington*

Is there a succession of digits which represents a prime in all systems of notation of radix r, $2 \leqq r \leqq 10$? If so, find the smallest one.

Note by E. P. Starke, Rutgers University. It will be shown that no such succession exists involving less than thirteen digits. Because of the prohibitive amount of labor involved in testing large numbers as possible primes, the solution will be carried no further.

Let the required numbers be represented by N_r. Since N_r is to have significance when $r=2$, its digits must be exclusively 1's and 0's, and we shall have

$$N_r = r^{a_1} + r^{a_2} + r^{a_3} + \cdots + r^{a_k},$$

where the $a_i (i = 1, 2, \cdots, k)$ are distinct non-negative integers and k is the number of digits 1. Evidently $a_k = 0$, since N_r is to be prime. Now no divisor $d \neq 1$ of k can be a divisor of $r-1$. For, if so, $r = dx+1$, whence each term r^{a_i} is of the form $dy+1$, and N_r, being the sum of k of these, is a multiple of d. Thus k must be at least eleven. However, N_r cannot contain as few as eleven digits (necessarily then all 1's) because if $N_r = (r^{11}-1)/(r-1)$ we have $N_4 = (2^{22}-1)/3$ which, by Fermat's theorem, is divisible by 23.

By special cases we show that N_r must have more than twelve digits. Suppose N_r contains eleven digits 1 and one digit 0. We then have

$$N_r = (r^{12} - 1)/(r - 1) - r^a, \qquad a = 1, 2, 3, \cdots, 10.$$

Now if $a=1$, 17 is a factor of N_5; $a=2$, 11 of N_4; $a=3$, 17 of N_3; $a=4$, 19 of N_3; $a=5$, 17 of N_2; $a=6$, 29 of N_2; $a=7$, 11 of N_4; $a=8$, 11 of N_2; $a=9$, 11 of N_7; $a=10$, 37 of N_2.

Construction of a Tetrahedron

E 709 [1946, 97]. *Proposed by N. A. Court, University of Oklahoma*

In a given sphere to inscribe a tetrahedron so that three concurrent edges shall pass through three given points and the plane of the three remaining edges shall be parallel to the plane determined by the three given points.

Solution by the Proposer. Let $DABC$ be the required tetrahedron whose edges DA, DB, DC pass through the given points P, Q, R, respectively. If the planes ABC and PQR are parallel, the two tetrahedrons $DABC$, $DPQR$ are homothetic, the vertex D being the homothetic center. Hence the sphere $DPQR$ is homothetic to the given circumsphere (O) of $DABC$, and the two spheres are tangent to each other at the homothetic center D. Thus the required vertex D is the point of contact of the given sphere (O) with a sphere belonging to the coaxal pencil of spheres passing through the circle determined by the three given points P, Q, R. Hence D may be found (*cf.* N. A. Court's *Modern Pure Solid Geometry*, art. 599). The problem may have two solutions.

Editorial Note. Recently W. A. Rees sent to this department the problem: In a given circle to inscribe a triangle so that two sides shall pass through two given points and the third side shall be parallel to the line determined by the two given points.

This problem is the two dimensional analogue of the above, and its solution is analogous to the solution of the above.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4215. *Proposed by Hüseyin Demir, Columbia University*

Prove that the Hermite polynomials defined as follows

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2},$$

have the property

$$n! \sum_{p=0}^{n} \frac{H_p^2(x)}{p!} = H_{n+1}^2(x) - H_n(x) H_{n+2}(x).$$

4216. *Proposed by Herbert Robbins, Annapolis, Md.*

Write $\{x_n\} \overset{c}{\to} \alpha$ if $\lim_{n\to\infty}(x_1+\cdots+x_n)/n=\alpha$. A function $f(x)$ is said to be *C. c.*, (Cesàro continuous) at $x=\alpha$ if $\{x_n\} \overset{c}{\to} \alpha$ implies $\{f(x_n)\} \overset{c}{\to} f(\alpha)$. Show that if $f(x)$ is of the form $Ax+B$ then it is *C. c.* at every value of x, and that if $f(x)$ is *C. c.* at even a single value $x=\alpha$, then $f(x)$ is of the form $Ax+B$.

4217. *Proposed by P. A. Piza, San Juan, P. R.*

Let a and b be positive integers whose sum is the square c^2 and whose difference is the cube d^3. Given that each of c^2, d^3, and a is a four digit integer and that the sum of the four digits contained in each is equal to b, find a and b.

4218. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $T\equiv ABCD$, if a point L with the normal coordinates (x, y, z, t) is such that its associates $(-x, y, z, t)$, $(x, -y, z, t)$, $(x, y, -z, t)$, $(x, y, z, -t)$ are on the circumsphere, it coincides with the point whose distances to the planes of the faces BCD, CDA, DAB, ABC are proportional to the radii of the circumcircles of these faces (second Lemoine point for T), and conversely.

4219. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In an orthocentric tetrahedron $ABCD$ with the altitudes AA', BB', CC', DD', let H' be the inverse of the orthocenter H with respect to the circumsphere, which the lines $H'A$, $H'B$, $H'C$, $H'D$ meet again in A_1, B_1, C_1, D_1. Show that the tetrahedrons $A_1B_1C_1D_1$ and $A'B'C'D'$ are similar and that the volume of the first is 27 times that of the second.

SOLUTIONS

Bound for a Finite Sum

4155 [1945, 220]. *Proposed by G. W. Wishard, Norwood, Ohio*

Find a formula for the sum of the series

$$1 + 2^2 + 3^3 + \cdots + n^n.$$

If such a formula is impossible find superior and inferior limits for the sum.

Solution by F. Underwood, University College, Nottingham. Let S_n denote the sum of the given series whose νth term is denoted by u_ν, then a first (crude) upper limit for S_n is given as follows,

$$S_{n-1} < (n-1)u_{n-1} = (n-1)^n u_n; \qquad S_n = S_{n-1} + u_n < 2u_n.$$

The first few cases show the crudity of this limit; thus $u_4=256$, $S_4=288$; $u_5=3125$, $S_5=3413$; and so a much closer inequality can be expected. Now

$$\frac{u_{n-1}}{u_n} = \frac{(n-1)^{n-1}}{n^n} = \frac{1}{n-1}\left(1-\frac{1}{n}\right)^n = \frac{v_n}{n-1}, \qquad v_n = \left(1-\frac{1}{n}\right)^n,$$

$$v_n - v_{n-1} = \left(1-\frac{1}{n}\right)^n - \left(1-\frac{1}{n-1}\right)^{n-1}$$

$$= \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^{n-1} - \left(1-\frac{1}{n-1}\right)^{n-1},$$

$$= \left(1-\frac{1}{n}\right)\left(1+\frac{1}{n-1}\right)^{-(n-1)} - \left(1-\frac{1}{n-1}\right)^{n-1}$$

$$= \left(1+\frac{1}{n-1}\right)^{-(n-1)}\left\{\left(1-\frac{1}{n}\right) - \left(1-\frac{1}{(n-1)^2}\right)^{n-1}\right\}.$$

For $n>1$

$$\left(1-\frac{1}{(n-1)^2}\right)^{n-1} < \left(1-\frac{1}{(n-1)^2}\right) < 1-\frac{1}{n}.$$

Hence $v_n - v_{n-1} > 0$ for $n>1$. Also $v_2 = 1/4$ and $\lim_{n\to\infty} v_n = 1/e$. Hence

$$\frac{1}{4} = v_2 < v_3 < v_4 < \cdots < v_n < \frac{1}{e},$$

$$S_{n-1} < 2u_{n-1} < \frac{2u_n}{e(n-1)},$$

$$S_n = S_{n-1} + u_n < u_n\left[1+\frac{2}{e(n-1)}\right], \quad \text{for} \quad n>1.$$

Again

$$u_{n-1} > \frac{u_n}{4(n-1)}, \quad \text{for} \quad n>2,$$

$$S_n = S_{n-1} + u_n > u_{n-1} + u_n > u_n\left[1+\frac{1}{4(n-1)}\right].$$

Hence, for $n>2$,

$$n^n\left[1+\frac{1}{4(n-1)}\right] < S_n < n^n\left[1+\frac{2}{e(n-1)}\right].$$

Editorial Note. The proposer gave the inequalities $n^n < S_n < 2n^n$. He also gave a table for the first ten values of S_n: it will suffice here to give the following extracts:

$$n = \quad 6, \qquad 7, \qquad 8, \qquad 9$$

$$S_n = 50069, \quad 873612, \quad 17650828, \quad 405071317.$$

The table suggested to him $n^n + n^{(n-1)}/2$ *might* be taken as a superior limit. A third, incomplete, solution uses the inequalities

$$\frac{n!e^n}{\sqrt{2\pi n}}\,\frac{12n-1}{12n} < n^n < \frac{n!e^n}{\sqrt{2\pi n}},$$

which are modifications of inequalities in Serret's *Algebra*, vol. 2, p. 249, §391.

Perfect Squares

4157 [1945, 220]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the base such that a number of eight digits of the form *ababcdcd* can be the square of a number of four digits *mnmn*, where the numbers of two digits *ab* and *cd*, or *ab* and *mn*, or *cd* and *mn* are consecutive.

Solution by E. P. Starke, Rutgers University. Put B for the base and A, C M for the two-digit numbers *ab*, *cd*, *mn*, respectively. Then the hypothesis gives

$$A(B^6 + B^4) + C(B^2 + 1) = M^2(B^2 + 1)^2,$$

or

$$A(B^2 - 1) + (A + C)/(B^2 + 1) = M^2.$$

The integer $(A+C)/(B^2+1)$ must have the value unity, so that

$$A(B^2 - 1) = M^2 - 1, \tag{1}$$

$$A + C = B^2 + 1. \tag{2}$$

(a) Suppose A and M are consecutive. Then $M=A+1$. ($M=A-1$ is inconsistent with $M<B^2$ and (1).) Hence (1) becomes $M=B^2-2$, and at once $A=B^2-3$, $C=04$. Thus there are squares of the proposed form for every base $B>4$. *e.g.* when $B=10$, $9898^2=97970404$.

(b) Suppose $A=C\pm1$, reducing (2) to $2A\pm1=B^2+1$. We may eliminate A between this equation and (1) with the result

$$2M^2 = B^4 + B^2 \quad \text{or} \quad 2M^2 = B^4 - B^2 + 2.$$

The first of these is impossible since by (2) B is even, and hence M^2 must contain 2 as a factor to an odd power. For the second, solutions are possible: the simplest is $B=4$, $M=11$. Thus, in the system of base 4, $2323^2=20202121$.

(c) Suppose $M=C\pm1$. Using equations (1), (2) and eliminating A and C, we may put the results in the form

$$(5B^2 - 3)^2 + 16 = 5(2M + B^2 - 1)^2 \quad \text{or} \quad (5B^2 + 1)^2 - 16 = 5(2M + B^2 - 1)^2.$$

For the first there seems to be no simple integral solution, but the second yields $B=7$, $M=31$. Thus in the system of base 7 we have $4343^2=26264242$.

Solved also by the proposer.

Editorial Note. The proposer gave three examples, the first two of which are above: base 4, $(2323)^2$; base 7, $(4343)^2$ base 7, $(6565^2)=64640404$.

No derivation was given.

A Partial Differential Equation

4158 [1945, 280]. *Proposed by P. D. Thomas, Lumberton, Miss.*

Integrate the partial differential equation

$$(1+q^2)r - 2pqs + (1+p^2)t + (rt-s^2)/(1+p^2+q^2)^{1/2} + (1+p^2+q^2)^{3/2} = 0;$$

and give a geometrical interpretation of the general integral.

Solution by C. E. Springer, University of Oklahoma. This non-uniform equation is of the form

$$Rr + Ss + Tt + U(rt - s^2) = F. \tag{1}$$

It can be solved by the well known method of Menge (see, for instance, Miller, *Partial Differential Equations*, N. Y., p. 222).

Substitution of

$$r = \frac{dp - sdy}{dx}, \qquad t = \frac{dq - sdx}{dy}$$

into equation (1) gives

$$Rdydp + Tdxdq + Udpdq - Fdxdy = s[R(dy)^2 - Sdxdy + T(dx)^2 + U(dxdp - dydq)]. \tag{2}$$

We regard equation (2) as an identity in s, and seek solutions of

$$\begin{aligned} Rdydp + Tdxdq + Udpdq - Fdxdy &= 0, \\ R(dy)^2 - Sdxdy + T(dx)^2 + U(dxdp + dydq) &= 0. \end{aligned} \tag{3}$$

It can be shown (see reference above) that a value of λ for which

$$R(dy)^2 - Sdxdy + T(dx)^2 + U(dxdp + dydq) + \lambda(Rdydp + Tdxdq + Udpdq - Fdxdy) \tag{4}$$

is factorable, is given by

$$(RT + FU)\lambda^2 + SU\lambda + U^2 = 0. \tag{5}$$

For the given differential equation, the roots of (5) are coincident and are expressed by

$$\lambda = \frac{1}{pq\sqrt{1+p^2+q^2}}. \tag{6}$$

The factors of the expression (4) are found to be

$$\begin{aligned} &Udy + \lambda Tdx + \lambda Udp, \\ &\lambda Rdy + Udx + \lambda Udq. \end{aligned} \tag{7}$$

On using the value of λ in (6) and the coefficients of the given differential equation, there results

$$\begin{aligned}\sqrt{1+p^2+q^2}\,[dx+pdz]+dp&=0,\\ \sqrt{1+p^2+q^2}\,[dy+qdz]+dq&=0.\end{aligned}\tag{8}$$

Use of $dz=pdx+qdy$ in equations (8) leads to the integrals

$$\begin{aligned}p&=(c_1-x)\sqrt{1+p^2+q^2}\\ q&=(c_2-y)\sqrt{1+p^2+q^2}.\end{aligned}\tag{9}$$

The values of p and q from equations (9), can be substituted into $dz=pdx+qdy$ to yield

$$dz[1-(x-c_1)^2-(y-c_2)^2]^{1/2}=(c_1-x)dx+(c_2-y)dy,$$

which can be integrated to give

$$(x-c_1)^2+(y-c_2)^2+(z-c_3)^2=1.$$

This equation represents all unit spheres in three-dimensional space.

Solved also by F. Underwood and the proposer.

Editorial Note. The solution by the proposer is similar to the above. Underwood stated that this problem is Ex. 2, p. 160 in the *Supplementary Volume* to Boole's *Treatise on Differential Equations*, which was revised by Todhunter. He gave a brief sketch of the solution, using the notation of *Differential Equations* by Piaggio, pp. 183, 184.

Determinants of Determinants

4159 [1945, 280]. *Proposed by F. J. Duarte, Caracas, Venezuela*

Given 18 numbers a_i, b_i, c_i, $i=1, 2, 3, 4, 5, 6$ such that

$$\begin{vmatrix}(bc)_{13}&(ca)_{13}&(ab)_{13}\\(bc)_{25}&(ca)_{25}&(ab)_{25}\\(bc)_{46}&(ca)_{46}&(ab)_{46}\end{vmatrix}=\begin{vmatrix}(bc)_{24}&(ca)_{24}&(ab)_{24}\\(bc)_{16}&(ca)_{16}&(ab)_{16}\\(bc)_{35}&(ca)_{35}&(ab)_{35}\end{vmatrix}$$

where

$$(mn)_{ij}=\begin{vmatrix}m_i&m_j\\n_i&n_j\end{vmatrix},$$

show that

$$\begin{vmatrix}(bc)_{12}&(ca)_{12}&(ab)_{12}\\(bc)_{34}&(ca)_{34}&(ab)_{34}\\(bc)_{56}&(ca)_{56}&(ab)_{56}\end{vmatrix}=0.$$

Solution by John Williamson, Queens College. Let (ijk) denote the determinant $|a_ib_jc_k|$ and $|(ij), (kl), (mn)|$ the determinant $|(bc)_{ij}(ca)_{kl}(an)_{mn}|$.

then

$$|(13), (25), (46)|(145) = \begin{vmatrix} 0 & (413) & (513) \\ (125) & (425) & 0 \\ (146) & 0 & (546) \end{vmatrix}$$

$$= -(413)(125)(546) - (513)(425)(146). \tag{i}$$

In the same way we find that

$$|(24), (16), (35)|(145) = -(524)(416)(135) - (124)(516)(435) \tag{ii}$$

and

$$|(12), (34), (56)|(145) = (412)(534)(156) + (512)(134)(456). \tag{iii}$$

Since by hypothesis both (i) and (ii) are zero,

$$(134)(125)(456) = -(135)(245)(146) = -(124)(156)(345)$$

and this implies that (iii) is zero. Therefore the required result is proved if $(145) \neq 0$. In a similar manner the result is true if $(236) \neq 0$.

Let $(145) = (236) = 0$. We may therefore assume that one of the vectors, say (a_1, b_1, c_1) is a linear combination of the other two (a_4, b_4, c_4) and (a_5, b_5, c_5). Accordingly we may write

$$(a_1, b_1, c_1) = \rho(a_4, b_4, c_4) + \sigma(a_5, b_5, c_5)$$

and

$$(a_2, b_2, c_2) = \lambda(a_3, b_3, c_3) + \mu(a_6, b_6, c_6).$$

Then

$$|(13), (25), (46)| = |\rho(43) + \sigma(53), \lambda(35) + \mu(65), (46)|$$

$$= \rho\lambda|(43), (35), (46)| + \rho\mu|(43), (65), (46)| + \sigma\mu|(53), (65), (46)|, \tag{iv}$$

$$|(24), (16), (35)| = \rho\lambda|(34), (46), (35)| + \lambda\sigma|(34), (56), (35)| + \mu\sigma|(64), (56), (35)| \tag{v}$$

and

$$|(12), (34), (56)| = \rho\mu|(46), (34), (56)| + \lambda\sigma|(53), (34), (56)|. \tag{vi}$$

If $(iv) = (v) = 0$, (vi) is also zero.

Geometrically the result is well known. If a_i, b_i, c_i are the homogeneous coordinates of six points P_i, the hypothesis states that P_1P_3, P_2P_5, P_4P_6 are concurrent and that P_2P_4, P_1P_6, P_3P_5 are also concurrent. These two sets of three lines determine two projective pencils and therefore P_1P_2, P_3P_4, P_5P_6 are concurrent.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Princeton University announces the appointment of Professor Emil Artin of Indiana University to a professorship and the promotion of Associate Professors Salomon Bochner, Claude Chevalley, and A. W. Tucker to professorships.

Syracuse University announces the appointment of Assistant Professor S. S. Cairns of Queens College to the chairmanship of the department of mathematics, the promotion of Associate Professor Charles Loewner to a professorship, and the appointment to assistant professorships of Dr. Roy Dubisch, Dr. D. E. Kibbey and Dr. P. C. Rosenbloom.

The University of California at Los Angeles announces the appointment to a professorship of Professor I. S. Sokolnikoff of the University of Wisconsin, the promotion to a professorship of Associate Professor E. F. Beckenbach, and the promotion to an associate professorship of Assistant Professor P. G. Hoel.

Dr. A. A. Aucoin has been appointed to an associate professorship at the University of Houston.

Dr. W. D. Berg has been appointed to a visiting assistant professorship at Kenyon College.

Assistant Professor D. H. Blackwell of Howard University has been promoted to an associate professorship.

Dr. Archie Blake has been appointed senior statistician in the Office of the Army Surgeon General, Washington, D. C.

Assistant Professor W. E. Bleick of the Postgraduate School, United States Naval Academy, has been promoted to an associate professorship.

Associate Professor Richard Brauer of the University of Toronto has been promoted to a professorship.

Dr. H. K. Brown of Lehigh University has been promoted to an assistant professorship of mechanics.

Dr. A. W. Burks of Swarthmore College has been appointed to an assistant professorship in philosophy at the University of Michigan.

Dr. J. D. Campbell of Rensselaer Polytechnic Institute has been promoted to an assistant professorship.

Dr. D. E. Christie of Bowdoin College has been promoted to an assistant professorship.

Dr. Paul Civin of the University of Buffalo has been appointed to an assistant professorship at the University of Oregon.

Dr. Robert Cortell of the College of the City of New York has been promoted to an assistant professorship.

Dr. V. F. Cowling of Ohio State University has been appointed to an assistant professorship at Lehigh University.

Dr. J. T. Culbertson has been appointed to an assistant professorship in philosophy at Michigan State College.

W. J. A. Culmer has been appointed to an assistant professorship at Hamline University, St. Paul, Minnesota.

Dr. A. B. Cunningham has been appointed to an assistant professorship at West Virginia University.

Assistant Professor John DeCicco of the Illinois Institute of Technology has been appointed to a visiting associate professorship at Columbia University.

Associate Professor C. H. Denbow of Ohio University, Athens, Ohio, has been appointed to an associate professorship at the Postgraduate School, United States Naval Academy.

Associate Professor W. J. Dixon of the University of Oklahoma has been appointed to an associate professorship at the University of Oregon.

Dr. D. M. Dribin has been appointed research analyst with the Army Security Agency, Washington, D. C.

Dr. J. E. Eaton of Queens College, Flushing, New York, has been appointed mathematician at the Naval Research Laboratory.

W. R. Eikelberger has been appointed to an assistant professorship at the University of Denver.

Dr. Herbert Federer of Brown University has been promoted to an assistant professorship.

Assistant Professor D. G. Fulton of the University of New Hampshire has been appointed to an associate professorship at Tufts College.

Dr. B. H. Gere has been appointed to an assistant professorship at the Postgraduate School, United States Naval Academy.

Dr. David Gilbarg, formerly at the Naval Ordnance Laboratory, has been appointed to an assistant professorship at Indiana University.

Assistant Professor B. E. Gillam of Drake University has been promoted to an associate professorship.

Assistant Professor J. W. Givens of Northwestern University has been appointed to an associate professorship at Illinois Institute of Technology.

Dr. L. J. Green of Case School of Applied Science has been promoted to an assistant professorship.

Assistant Professor P. R. Halmos of Syracuse University has been appointed to an assistant professorship at the University of Chicago.

Assistant Professor R. W. Hamming of the University of Louisville has been appointed a member of the technical staff at Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. E. A. Hedberg has been appointed to an associate professorship at the University of South Carolina.

Assistant Professor A. E. Heins of Purdue University has been appointed to an associate professorship at the Carnegie Institute of Technology.

Dr. R. G. Helsel of Ohio State University has been promoted to an assistant professorship.

Dr. J. G. Herriot of Stanford University has been promoted to an assistant professorship.

Dr. Edwin Hewitt of the Institute for Advanced Study has been appointed to an assistant professorship at Bryn Mawr College.

Dr. F. E. Hohn has been appointed to an assistant professorship at the University of Maine.

Dr. P. L. Hsu has been appointed to an associate professorship at the University of North Carolina.

W. N. Huff of the University of Rochester has been appointed to an assistant professorship at the University of Oklahoma.

Assistant Professor P. M. Hummel of the University of Alabama has been promoted to an associate professorship.

Dr. H. D. Huskey of the University of Pennsylvania has been appointed to an assistant professorship at the University of Oklahoma.

Associate Professor B. O. Koopman of Columbia University has been promoted to a professorship.

Associate Professor R. L. Krueger of Wittenberg College has been promoted to a professorship.

Dr. Mary E. Ladue of Barnard College, Columbia University, has been promoted to an assistant professorship.

Dr. J. P. LaSalle has been appointed to an assistant professorship at the University of Notre Dame.

T. H. Lee of Louisiana State University has been appointed to an adjunct professorship at the University of South Carolina.

Associate Professor D. C. Lewis of the University of New Hampshire has been appointed to a professorship at the University of Maryland.

Assistant Professor C. C. Lin of Brown University has been promoted to an associate professorship.

Professor C. E. Miller of the University of New Brunswick has been appointed to a professorship at the University of Saskatchewan.

R. A. Miller has been appointed to an assistant professorship at the University of Mississippi.

W. B. Morgan has been appointed to an assistant professorship at Bluffton College, Bluffton, Ohio.

Dr. H. T. Muhly of the United States Naval Academy has been promoted to an assistant professorship.

Associate Professor W. R. Murray of Franklin and Marshall College, Lancaster, Pennsylvania, has been promoted to a professorship.

Dr. A. M. Peiser has been appointed to an assistant professorship at Rutgers University.

Dr. C. R. Phelps has been appointed to an assistant professorship at Rutgers University.

Dr. R. S. Phillips of the Massachusetts Institute of Technology has been appointed to an assistant professorship at New York University.

Dr. Harry Pollard of Yale University has been appointed to an assistant professorship at Cornell University.

Associate Professor R. G. Putnam of New York University has been promoted to a professorship.

Associate Professor G. E. Raynor of Lehigh University has been promoted to a professorship.

E. K. Ritter has been appointed to an assistant professorship at the Postgraduate School, United States Naval Academy.

Dr. D. H. Rock has been appointed to an assistant professorship at Iowa State College.

Dr. Robert Schatten of the Institute for Advanced Study has been appointed to an associate professorship at the University of Kansas.

Associate Professor C. H. W. Sedgewick of the University of Connecticut has been promoted to a professorship.

Dr. J. K. Senior of the University of Chicago has been promoted to an associate professorship.

Dr. C. B. Smith has been appointed to an associate professorship at the University of Florida.

Professor C. E. Springer of the University of Oklahoma has been appointed chairman of the department of mathematics.

Assistant Professor I. L. Stright of Baldwin-Wallace College has been appointed associate professor of mathematics and chairman of the department at Northern Michigan College of Education.

Dr. A. C. Sugar of Brown University has been appointed to an associate professorship at Oklahoma Agricultural and Mechanical College.

Dr. Feodor Theilheimer of Trinity College, Hartford, Connecticut, has been promoted to an assistant professorship.

Assistant Professor C. B. Tucker of Kansas State Teachers College of Emporia has been promoted to an associate professorship.

Assistant Professor R. J. Walker of Cornell University has been promoted to an associate professorship.

Professor H. S. Wall of the Illinois Institute of Technology has been appointed to a professorship at the University of Texas.

Dr. W. R. Wasow of New York University has been appointed to an assistant professorship at Swarthmore College.

Associate Professor Max Zorn of the University of California at Los Angeles has been appointed to a professorship at the University of Indiana.

The following appointments to instructorships are announced:
Brown University: Winfield Keck
Columbia University: Dr. T. W. Anderson, J. L. Bogdanoff
Illinois Institute of Technology: Dr. Margaret S. Matchett
Johns Hopkins University: Dr. Murray Mannos, Emanuel Mehr
Mount Holyoke College: Grace E. Bates
Rutgers University: Dr. H. J. Zimmerberg
Syracuse University: Dr. P. W. Gilbert, Dr. Ruth E. Goodman, Kathryn Morgan
United States Naval Academy: K. F. McLaughlin
University of California at Los Angeles: Dr. Leonard Greenstone
University of Michigan: Dr. W. F. Eberlein, Dr. Leonard Tornheim
University of Virginia: V. L. Klee
University of Wisconsin: Dr. R. E. Fullerton

Professor Federigo Enriques of the University of Rome died June 14, 1946.

Assistant Professor A. S. McMaster of the University of Colorado died May 24, 1946.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE PUTNAM MATHEMATICAL COMPETITION FOR 1946

GARRETT BIRKHOFF, Harvard University

The following results of the sixth annual William Lowell Putnam Mathematical Competition held June 1, 1946, have been determined in accordance with the rules of the Competition agreed to by the representatives of the Mathematical Association and the trustees of the William Lowell Putnam Intercollegiate Memorial Fund. The contestants were known to Association officials and to the reader only by number up to the time of this announcement. A total of sixty-seven undergraduate students representing seventeen institutions took part in the Competition.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Canada. The members of the team were Donald Fraser, Harry Seigel and James Stanley; to each of these, a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of the Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were Felix Browder, Eugene Calabi and A. A. Kheiralla; to each of these, a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Brooklyn College, Brooklyn, New York. The members of the team were Julian Keilson, Gerald Washnitzer and Arthur Zeichner; to each of these, a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of Carnegie Institute of Technology, Pittsburgh, Pennsylvania. The members of the team were Louis Bright, George Hinman and Julia Randall; to each of these, a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, were: Felix Browder, Massachusetts Institute of Technology; Eugene Calabi, Massachusetts Institute of Technology; Donald Fraser, University of Toronto; J. Arthur Greenwood, Harvard University; Maxwell Rosenlicht, Columbia University. Each of these will receive a prize of fifty dollars.

The five persons ranking next highest in the examination, named in alphabetical order, were: G. W. Hinman, Carnegie Institute of Technology; H. O. Seigel, University of Toronto; W. T. Sharpe, University of Toronto; J. P. Stanley, University of Toronto; Arthur Zeichner, Brooklyn College. Each of these will receive a prize of twenty dollars.

The following teams won honorable mention: Department of Mathematics, Columbia University, the members of the team being Alexander Heller, Alan Hoffman and Maxwell Rosenlicht; Department of Mathematics, Cornell University, the members of the team being J. E. Hanson, J. B. Hudders and J. E. Storer.

Five individuals are given honorable mention. The names, listed in alphabetical order, are: Julian Keilson, Brooklyn College; Leroy Meyers, Queens College; Norman Nadler, McGill University, Montreal; John Nash, Jr., Carnegie Institute of Technology; L. P. Shapley, Harvard University.

The following is a list, arranged alphabetically, of all colleges and universities which entered teams in the Competition, some also entering individual contestants: Brooklyn College, Brown University, University of California at Los Angeles, Carnegie Institute of Technology, Columbia College, Cornell University, Duke University, Harvard University, McGill University, Massachusetts Institute of Technology, Northwestern University, Stanford, Swarthmore College, University of Toronto. The following additional colleges and universities entered individual contestants only: University of Michigan; Queens College; Radcliffe College.

The Examination was given in two parts on the morning and afternoon of June 1, 1946. Contestants were allowed three hours upon each part. The questions follow.

PART I

1. Suppose that the function $f(x)=ax^2+bx+c$, where a, b, c are real constants, satisfies the condition $|f(x)| \leqq 1$ for $|x| \leqq 1$. Prove that $|f'(x)| \leqq 4$ for $|x| \leqq 1$.

2. If $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are polynomials in x, show that

$$\int_1^x a(x)c(x)\,dx \cdot \int_1^x b(x)d(x)\,dx - \int_1^x a(x)d(x)\,dx \cdot \int_1^x b(x)c(x)\,dx$$

is divisible by $(x-1)^4$.

3. A projectile in flight is observed simultaneously from four radar stations which are situated at the corners of a square of side b. The distances of the projectile from the four stations, taken in order around the square, are found to be R_1, R_2, R_3, R_4. Show that

$$R_1^2 + R_3^2 = R_2^2 + R_4^2.$$

Show also that the height h of the projectile above the ground is given by

$$h^2 = -\frac{1}{2}b^2 + \frac{1}{4}(R_1^2 + R_2^2 + R_3^2 + R_4^2)$$
$$-\frac{1}{8b^2}(R_1^4 + R_2^4 + R_3^4 + R_4^4 - 2R_1^2R_3^2 - 2R_2^2R_4^2).$$

4. Let $g(x)$ be a function that has a continuous first derivative $g'(x)$ for all values of x. Suppose that the following conditions hold: (i) $g(0)=0$. (ii) $|g'(x)| \leqq |g(x)|$ for every x. Prove that $g(x)$ vanishes identically.

5. Find the smallest volume bounded by the coordinate planes and by a tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

6. A particle of unit mass moves on a straight line under the action of a force which is a function $f(v)$ of the velocity v of the particle, but the form of this function is not known. A motion is observed, and the distance x covered in time t is found to be connected with t by the formula $x=at+bt^2+ct^3$, where a, b, c have numerical values determined by observation of the motion. Find the function $f(v)$ for the range of v covered by the experiment.

PART II

1. Let K denote the circumference of a circular disc of radius one, and let k denote a circular arc that joins two points a, b on K and lies otherwise in the given circular disc. Suppose that k divides the circular disc into two parts of equal area. Prove that the length of k exceeds 2.

2. Let A, B be variable points on a parabola P, such that the tangents at A and B are perpendicular to each other. Show that the locus of the centroid of the triangle formed by A, B and the vertex of P is a parabola P_1. Apply the same process to P_1, obtaining a parabola P_2, and repeat the process, obtaining altogether the sequence of parabolas P, P_1, P_2, $\cdots$, P_n. If the equation of P is $y^2=mx$, find the equation of P_n.

3. In a solid sphere of radius R the density ρ is a function of r, the distance from the center of the sphere. If the magnitude of the gravitational force of attraction due to the sphere at any point inside the sphere is kr^2, where k is a constant, find ρ as a function of r. Find also the magnitude of the force of attraction at a point outside the sphere at a distance r from the center. (Assume that the magnitude of the force of attraction at a point P due to a thin spherical shell is zero if P is inside the shell, and is m/r^2 if P is outside the shell, m being the mass of the shell, and r the distance of P from the center.)

4. For each positive integer n, put

$$p_n = (1 + 1/n)^n, \qquad P_n = (1 + 1/n)^{n+1}, \qquad h_n = \frac{2p_nP_n}{p_n + P_n}.$$

Prove that $h_1<h_2<\cdots<h_n\cdots$.

5. Show that the integer next above $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}.

6. A particle moves on a circle with center O, starting from rest at a point P and coming to rest again at a point Q, without coming to rest at any intermediate point. Prove that the acceleration vector of the particle does not vanish at any point between P and Q, and that, at some point R between P and Q, the acceleration vector points in along the radius RO.

THE SUPERVISION OF TEACHING ASSISTANTS

B. W. Jones, Cornell University

Since, in recent years, the department of mathematics at Cornell University has found it necessary to employ an increasing number of teaching assistants (graduate students who teach part time) members of the staff decided to create some plan for the systematic supervision of this teaching with a view to contributing as much as possible towards its improvement. Since inquiry revealed that other institutions have considered the same problem, letters were written to fifteen colleges and universities requesting information; replies were received from all but two. This is a brief digest of these replies.

Of the four institutions having no supervision, two had substantially no teaching assistants; the other two felt a need for supervision but had been prevented by lack of time and staff from formulating any plan of action. Six departments report that they have, or plan to have shortly, some visiting of the classes of assistants by regular members of the staff. In one of the six there is visiting only when there are complaints from students or other sources. At the other extreme, in another institution, formal visits are made on an assistant's

classes; the visitor then confers with the assistant, gives him a grade on his teaching and reports his findings to the other members of the permanent staff. In three of the institutions a regular member of the staff is in charge of each course involving sections taught by assistants, and this staff member, from time to time, calls meetings of those teaching the course and helps the assistants with their teaching problems. A uniform final examination seems to be common for most elementary courses taught in several sections, but it does not seem to be the general practice to give uniform tests during the term. In one university a member of the staff is given the general responsibility of supervising all teaching assistants and he, together with other members of the staff, visits classes taught by assistants, and assists them with their problems.

There seems to be a strong belief that there should be some supervision of beginning teachers, not only on the part of the members of the departments consulted, but among our own teaching assistants who have very definite ideas on the subject. This author would welcome suggestions from mathematicians who have had experience with the supervision of teaching assistants.

APPEAL FOR AID TO DEVASTATED LIBRARIES

About a year ago the Council of the Society appointed a committee charged with recommending a policy for bringing to libraries devastated by the war such aid as it would be possible for mathematicians of this country to give. The committee consists at present of Professors Garrett Birkhoff, H. E. Bray, Arnold Dresden, William Feller, Gabor Szegö, and A. W. Tucker. Recently the Board of Governors of the Association voted to cooperate with this committee in such policies as they might suggest.

A great deal of time and thought has been spent by the committee in obtaining information concerning the status of mathematical libraries abroad, and in making contact with agencies, private and semi-governmental, which have been created for similar purposes. It has become clear that the need is very great, particularly for books and periodicals published during the years 1939 to 1946. A call must be made on individual mathematicians for assistance.

Accordingly, an appeal is now being made for mathematicians to contribute books, journals, and reprints to a collection to be sent to devastated libraries. Please send only material in good condition. Send by express or parcel post, prepaid and addressed as follows:

American Mathematical Society
Butler Library, Columbia University
New York 27, New York
FOR FOREIGN LIBRARIES

It is important not to omit any part of this address.

Contributions in cash, made out to the American Mathematical Society, should be sent to the Society, 531 West 116th Street, New York 27, New York, with the notation "For Foreign Libraries."

THE MATHEMATICAL ASSOCIATION OF AMERICA

APRIL MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held at North Carolina State College, Raleigh, N.C., on Friday and Saturday April 19–20, 1946.

There were one hundred and five present, including the following forty-six members of the Association: Louise Adams, S. R. Baker, Helen Barton, A. T. Brauer, E. T. Browne, R. C. Bullock, E. A. Cameron, C. L. Carroll, Jr., T. C. Carson, J. W. Cell, Helen E. Clarkson, J. B. Coleman, Nelle C. Douglas, F. G. Dressel, L. A. Dye, W. W. Elliott, Tomlinson Fort, L. L. Garner, J. J. Gergen, C. L. Hair, Minnie W. C. Harris, Archibald Henderson, T. F. Hickerson, M. A. Hill, Jr., J. W. Lasley, Jr., Jack Levine, Anne L. Lewis, F. A. Lewis, R. S. Pate, K. B. Patterson, G. E. Reves, J. O. Reynolds, L. V Robinson, C. L. Seebeck, Jr., C. Eucebia Shuler, W. H. Spragens, Ruth W. Stokes, C. F. Strobel, Cora Strong, Mary M. Templeton, J. M. Thomas, G. C. Watson, G. T. Whyburn, W. L. Williams, Y. K. Wong, H. J. Zimmerberg.

Chairman J. W. Lasley, Jr. presided at the Friday afternoon and Saturday morning meetings, and Professor H. A. Fisher, North Carolina State College, presided on Friday evening at the informal dinner given in honor of the visiting speaker, Professor G. T. Whyburn. In the absence of the Secretary, Colonel H. A. Robinson, Professor Ruth W. Stokes acted as Secretary.

At the business session the following officers were elected for the coming year: Chairman, Ruth W. Stokes, Winthrop College; Vice-chairman, J. W. Cell, North Carolina State College; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. W. L. Williams, University of South Carolina, was elected Governor of Region V for two years. The Section voted to hold its 1947 meeting at Winthrop College. Resolutions were passed relative to the passing of Professor W. P. Ott who had held the position of Section Chairman three terms.

The program consisted of the following papers:

1. *On the exact number of primes below a given limit*, by Professor A. T. Brauer, University of North Carolina.

The speaker gave a simple proof of Meisell's formula for the exact number of primes below a given limit, and applied his results to some interesting examples.

2. *Projective flecnode ruled surfaces*, by Professor J. D. Novak, University of South Carolina.

The author gave the results of his study of two classes of ruled surfaces whose two flecnode surfaces are projectively equivalent. Cech's theorem and other generalizations were obtained by use of derivative curves in five dimensions.

3. *Some geometrical constructions of conics*, by Professor L. A. Dye, The Citadel.

Without resorting to the method of rotation of axes, theorems were stated which enable one to construct a conic in general position and to determine the equations and lengths of the axes. These were susceptible of proof by elementary methods not ordinarily introduced in a first analytic geometry course.

4. Retiring Chairman's Address: *The rule of the complex number in elementary mathematics*, by Professor J. W. Lasley, Jr., University of North Carolina.

A transition was outlined from the complex numbers of elementary algebra to the vector representation in the plane, with emphasis upon the concepts of addition and differentiation. Certain relations were formulated with analytic description, followed by geometric interpretation. These relations were applied to the proof of the sum and difference formulas of trigonometry, the rotation formulas of analytics, the plane curvilinear motion of the calculus, and the planetary motion of mechanics.

5. *Value distribution of a function of two complex variables in variables in domain with distinguished boundary surface*, by Professor Josephine Mitchell, Winthrop College, introduced by Professor Stokes.

Using the notion of distinguished boundary surfaces as introduced by S. Bergman, the author investigated meromorphic functions of two complex variables defined in a connected domain with distinguished boundary surface. Inequalities were obtained for various classes of complex functions and for the values which such functions could assume.

6. *Mathematical aspects of some problems in air navigation*, by Professor E. A. Cameron, University of North Carolina.

Certain navigation problems involving relative motion were formulated in terms of moving coördinate reference frames. The author demonstrated how such formulation helps to clarify the nature and significance of problems in air navigation.

7. *The rehabilitation of geometry*, by Professor W. W. Rankin, Duke University.

Professor Rankin stated that geometry is rapidly slipping out of the curriculum. He asserted we should do more to retain it or its equivalent.

8. *Mathematical program for college majors who plan to teach*, Professor Ruth W. Stokes, Winthrop College.

The speaker reviewed recent reports of national committees concerning mathematics in our post-war curriculum. She outlined certain preparatory courses for high school mathematics teachers, and included suggestions on the use of departmental clubs and multisensory aids.

9. *The area of a polygon when the sides and interior angles are known*, by Professor T. F. Hickerson, University of North Carolina.

10. *Surface topology and mappings*, by Professor G. T. Whyburn, University of Virginia.

Using Stokes's theorem as an example, the author showed how topological considerations arise in other branches of mathematics and how they contribute in lending definiteness to results in these fields. A discussion of orientability and Euler characteristic of surfaces was given, leading to the solution of the problem of classifying surfaces into topological types. It was then shown how this, together with the characterization of a surface as a topological space, yielded a complete and satisfactory solution to a most fundamental problem in surface topology. Professor Whyburn concluded with a brief discussion of interior mapping of surfaces and called attention to the as yet unsolved problem of classifying the possible interior images of surfaces of various types.

11. *The accuracy of linear interpolation*, by Professor P. M. Hummel, University of Alabama, read by Doctor Seebeck.

The purpose of this paper was to develop rather sharp upper and lower limits for the error resulting when a function with certain restrictions is evaluated between known values by linear interpolation.

12. *A coefficient of correlation for three variables*, by Doctor J. B. Coleman, Mathematician, S.C. State Highway Department.

In space of three dimensions the sum of the squares of the perpendicular distances of a set of points from a line of best fit was found. Associated with this sum was a formula containing the r's of Pearson's product moment coefficients by pairs. This expression measured the closeness of fit to a line, hence the degree of correlation for the three variables.

13. *A method in the theory of least squares*, by Doctor Y. K. Wong, University of North Carolina.

14. *Equivalent annuities*, by Professor C. L. Seebeck, Jr., University of Alabama.

The speaker proposed a treatment of annuities in the general case by means of equivalent simple annuities. The theory was extended to apply to problems of a type not usually treated in current textbooks. The interpolation theorem for finding the final payment of a simple annuity was generalized.

15. The equation $\ddot{x}+x=0$, by Professor W. H. Spragens, Georgia State Womans College.

Professor Spragens discussed the use of visual aids in teaching and listed sources of mathematical films now available. He showed an English film on the equation $\ddot{x}+x=0$.

16. *Analytic functions and interior transformations*, by Professor G. T. Whyburn.

It was shown first that an analytic function of a complex variable generates an interior transformation from the complex plane or sphere to the complex plane or sphere. This yielded a simple topological proof of the maximum modulus theorems, as well as results concerned with level lines and zeros of an analytic function. A discussion was then given of some recent results of Nevanlinna,

Ahlfors, and others concerning defect relations and totally branched values of meromorphic functions. It was shown also how in certain cases these same results may be obtained in a purely topological way using a special case of a characteristic equation for interior mappings from one surface to another.

17. *Some applications of differential operator methods*, by Professor L. V. Robinson, University of South Carolina.

A brief survey was made of the use of a differential operator in generalizing Taylor's formula and in solving certain types of differential equations.

18. *The solution of a differential equation as the limit of the solution of a difference equation*, by Professor Tomlinson Fort, University of Georgia.

Although the problem is old, it is believed that Professor Fort's novel treatment yielded results not previously known. The difference equations treated were equations in discrete argument (recurrent relations). They were set up in terms of a real parameter and the limit was approached by letting this parameter become infinite. The problem was then one in ordinary analysis. The work was carried out by forming a series for the solution of the difference equation whose terms approached terms of the corresponding series solution of the differential equation. Applications were made to the Sturm-Liouville theory.

19. *Definite differential and integral systems*, by Dr. H. J. Zimmerberg, North Carolina State College.

The author summarized the results obtained by Bliss and Reid for differential systems in matrix form under various definitions of definiteness. Then he discussed results for integral systems of Friedholm type under corresponding assumptions.

20. *Differential equations with quadrilateral envelope, cuspidal, and nodal loci*, by Professor Archibald Henderson, University of North Carolina.

This paper has appeared in the *National Mathematics Magazine*, Vol. XX, No. 2, pp. 1–18.

H. A. ROBINSON, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The thirtieth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on April 4, 1946. Professor J. B. Brandeberry, Chairman of the Section, presided.

Sixty-eight persons attended, including the following forty-three members of the Association: Grace M. Bareis, H. M. Beatty, Louis Brand, J. B. Brandeberry, O. E. Brown, V. B. Caris, Florentina M. Clinton, Wayne Dancer, H. P. Fawcett, M. P. Fobes, B.C. Glover, H. H. Hartzler, R. G. Helsel, R. C. Hildner, Margaret E. Jones, L. C. Knight, A. C. Ladner, H. E. Menke, E. J. Mickle, H. J. Miser, Max Morris, F. D. Murnaghan, C. V. Newsom, Emma J. Olson, Tibor Radó, J. F. Randolph, F. Mozelle Rankin, S. E. Rasor, Hortense Rickard, R. F. Rinehart, S. A. Rowland, K. C. Schraut, Samuel Selby, E. S. Smith,

H. E. Stelson, J. L. Synge, H. S. Toney, D. R. Whitney, R. B. Wildermuth, C. O. Williamson, C. R. Wylie, Jr., E. P. Vance, C. H. Yeaton.

The following officers were elected for the coming year: Chairman, S. A. Rowland, Ohio Wesleyan University; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, E. J. Mickle, Ohio State University; Program Committee, F. B. Wiley, Denison University, C. H. Yeaton, Oberlin College, Wayne Dancer, University of Toledo. It is expected that the next meeting will be held at Ohio State University on April 3, 1947.

The following program was presented:

1. *Imaginary branches of real curves*, by Professor J. B. Brandeberry, University of Toledo.

By introducing an iY axis through the origin and perpendicular to the XY plane, the imaginary branches of curves can be graphed in the same manner as the real branches. Intersections, maxima and minima, points of inflection, *etc.*, when imaginary, can be illustrated and explained by means of these graphs. The student gains a better understanding of "excluded values" and the real branches of curves after studying their behavior in the imaginary plane.

2. *Finding the term in annuity equations*, by Professor H. E. Stelson, Kent State University.

The speaker discussed the meaning of the term of an annuity when a fractional value is obtained by solving an annuity equation. The interpolated value can be assigned a meaning in both the present value and final value annuity formulas. Solution by logarithms gives a result which bears a relationship to the interpolated result.

3. *A computer for celestial navigation*, by Professor O. E. Brown, Case School of Applied Science.

Professor Brown described a computer consisting of a transparent sheet pivoted to an opaque sheet. The opaque sheet carries an azimuthal equidistant projection of the celestial sphere displaying a row of dots for each of forty-four navigational stars, each row containing one dot for each integer degree of local hour angle. The transparent sheet carries curves of constant azimuth and curves of constant altitude. Rotation of the transparent sheet corresponds to latitude. The assumed local hour angle and the name of the star select one dot on the opaque sheet. When the rotor is set to the assumed latitude, the appearance of the selected dot among the curves of the rotor permit the altitude and azimuth of the star to be estimated with amazing accuracy. Symmetry considerations and the use of both sides of the opaque sheet make it possible to employ only one quadrant of the otherwise circular figures.

4. *Motors as dual vectors*, by Professor Louis Brand, University of Cincinnati.

A vector $\mathbf{a}$ localized in a line $\mathbf{r}\times\mathbf{a}=\mathbf{a}_0$ may be specified by its Plücker coordinates $(\mathbf{a}, \mathbf{a}_0)$; these satisfy the relation $\mathbf{a}\cdot\mathbf{a}_0=0$. The *moment vector* depends upon the choice of 0. Referred to P the moment vector is given by the *shift formula* $\mathbf{a}_P=\mathbf{a}_0+\overrightarrow{PO}\times\mathbf{a}$.

The ∞^4 lines of space may be represented by Plücker coördinates for which $\mathbf{a}\cdot\mathbf{a}=1$, $\mathbf{a}\cdot\mathbf{a}_0=0$. These may be combined into a *dual* vector $\mathbf{A}=\mathbf{a}+\epsilon\mathbf{a}_0$, where the algebraic unit ϵ has the property $\epsilon^2=0$. If we expand $(\mathbf{a}+\epsilon\mathbf{a}_0)\cdot(\mathbf{a}+\epsilon\mathbf{a}_0)$ by applying the distributive law, $\mathbf{A}\cdot\mathbf{A}=1$. Thus the dual vector which represents a line is a *unit* vector.

The ∞^5 line vectors of space may be represented as real multiples of unit line vectors: $\mathbf{B}=\lambda(\mathbf{a}+\epsilon\mathbf{a}_0)$. The Plücker coördinates $\mathbf{b}=\lambda\mathbf{a}$, $\mathbf{b}_0=\lambda\mathbf{a}_0$ still satisfy $\mathbf{b}\cdot\mathbf{b}=0$ and $\mathbf{b}_P$ is computed from the shift formula.

The ∞^6 *motors* $\mathbf{M}$ of space are now defined as *dual* multiples of unit line vectors: $\mathbf{M}=(\lambda+\epsilon\lambda')\mathbf{A}$. Thus $\mathbf{M}=\lambda\mathbf{a}$, $\mathbf{M}_0=\lambda\mathbf{a}_0+\lambda'\mathbf{a}$; $\mathbf{M}_P$ is still given by the shift formula but $\mathbf{M}\cdot\mathbf{M}=\lambda\lambda'$. Motors may be classified as *screws* ($\lambda,\ \lambda'\neq 0$), *line vectors* ($\lambda\neq 0,\ \lambda'=0$), *couples* ($\lambda=0,\ \lambda'\neq 0$), or *zero* ($\lambda=0,\ \lambda'=0$).

The products $\mathbf{M}\cdot\mathbf{N}$ and $\mathbf{M}\times\mathbf{N}$ are now defined by applying the distributive law to the dual vectors. All rules of calculation for this motor algebra are in precise analogy with the rules of ordinary vector algebra. By means of the concept of *dual* angle, $\mathbf{M}\cdot\mathbf{N}$ and $\mathbf{M}\times\mathbf{N}$ may be expressed exactly as the corresponding vector products. Finally this algebra permits of reciprocal sets of motors which have all the properties of reciprocal sets of vectors.

5. *The teaching of college mathematics*, by Professor F. D. Murnaghan, Johns Hopkins University.

This paper appears in the present issue of the MONTHLY.

6. *A report of the committee on the teaching of elementary mathematics*, by Professor H. P. Fawcett, Ohio State University.

Requirements for the certification of elementary teachers in Ohio include, among other things, 24 semester hours of work in special subjects, 15 in the field of English, 8 in the field of science. There are no specific requirements in mathematics, although under the subject of arithmetic the current elementary teacher training curriculum prescribed in 1938 states that "the teacher in training shall possess mastery of the four fundamentals, common and decimal fractions, and percentage, including the special case of interest." Teacher training institutions are then asked to provide "guarantees that the graduates of the elementary teacher training program possess certain primitive knowledge and skills deemed essential to successful teaching."

To deal with this problem the teacher training institutions of Ohio are following a variety of practices, ranging all the way from a few weeks spent on the teaching of arithmetic to the use of a carefully prepared diagnostic test followed by six hours of required courses in professionalized subject matter.

Following a comprehensive study of the situation, this committee has proposed a tentative course of study in mathematics for prospective teachers in the elementary schools of Ohio. The plan has been submitted to the teacher training institutions for criticism and study. The committee asks for further development of this proposal, and appropriate action.

RUFUS CRANE, *Secretary*

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the U. S. Naval Academy, Annapolis, Maryland, on Saturday, December 8, 1945. Professor J. B. Scarborough, Chairman of the Section, presided at morning and afternoon sessions.

The attendance was one hundred, including the following fifty members of the Association: H. C. Ayres, R. A. Baumgartner, W. E. Bleick, H. H. Campaigne, Randolph Church, G. R. Clements, Abraham Cohen, J. F. Daly, Alexander Dillingham, J. A. Duerksen, P. J. Federico, E. J. Finan, B. H. Gere, Michael Goldberg, R. A. Good, J. R. Hammond, D. C. Harkin, L. M. Kells, D. M. Krabill, Solomon Kullback, R. C. Lamb, A. E. Landry, J. F. Locke, G. A. Lyle, Carol V. McCamman, E. S. Mayer, D. D. Miller, Helen K. Milleson, J. F. Milos, T. W. Moore, Eugenie M. Morenus, H. M. Nahikian, W. H. Norris, Jr., C. R. Phelps, Grace S. Quinn, C. H. Rawlins, Jr., C. E. Rhodes, J. N. Rice, R. E. Root, C. A. Rupp, J. B. Scarborough, E. D. Schell, Veryl G. Schult, Hillel Spitz, E. G. Swafford, O. M. Thomas, V. J. Varineau, R. W. Wagner, Hazel S. Wilson, R. H. Wilson, Jr.

The first four of the following papers were read at the morning session. The last paper was read at the afternoon session by Lieut. Comdr. J. F. Daly at the invitation of the Section.

1. *Magic rectangles modulo p*, by Professor E. J. Finan, Catholic University. This paper has since appeared in this MONTHLY.

2. *Approximation to an analytic function*, by Ensign E. N. Nilson, U.S.N.R., introduced by the Secretary.

A problem of the best approximation to an analytic function in the sense of least pth powers was discussed for the case of least squares with the function analytic in the unit circle of the complex plane. The approximating functions were restricted to be analytic within, and of bounded integral mean on, a larger circle. The measure of approximation was taken to be the integral mean, over a circle within the unit circle, of the modulus of the difference between the given function and the approximating function.

3. *On a set of almost orthogonal trigonometric functions*, by Lieut. R. P. Bailey, U.S.N.R., introduced by the Secretary.

The speaker discussed the properties of a set of trigonometric functions $\chi_m(x) = \cos \lambda_m x / \cos \lambda_m \pi$, $(m = 0, 1, 2, \cdots)$ arising in a boundary value problem connected with the heat equation. The characteristic numbers $\{\lambda_m\}$ appear as the roots of the equation $\tan \lambda\pi = \frac{1}{2}M\lambda$. It was shown that the non-harmonic "Fourier" expansion

$$\sum \chi_m(x) \left[\int_{-\pi}^{\pi} f(t)\chi_m(t)dt - Mf(\pi) \right] \cdot \left[\int_{-\pi}^{\pi} \chi_m^2(t)dt - M \right]^{-1}$$

for an arbitrary even periodic function $f(x)$ belonging to L^2 on $(-\pi, \pi)$ always converges to $f(\pm\pi)$ at the endpoints of the interval, and in the interior has exactly the same convergence (or summability) properties as the ordinary Fourier cosine expansion for the function.

4. *Mathematics in the Washington Veterans High School Center*, by Carol V. Coleman, Washington Veterans High School Center.

This paper dealt with some of the problems apt to confront any mathematics teacher whose class is at the high school or freshman college level and composed principally of veterans.

5. *Some applications of probability theory to the inspection of materials*, by Lieut. Comdr. J. F. Daly, U.S.N.R.

The speaker discussed the application of certain basic concepts of probability theory to the control of quality of manufactured articles, with particular emphasis on the effectiveness of rational methods of lot acceptance inspection. The notion of the operating characteristic curve of a sampling plan was explained, and was used to clarify the meaning of such quality control terms as average outgoing quality limit, acceptable quality level, consumer's risk, producer's risk, and tolerance percent defective. Illustrations were taken from the wartime experience of both industry and government to show the improvement in quality which can be achieved through the systematic use of even the most elementary statistical methods.

E. J. FINAN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirtieth Annual Meeting, Swarthmore, Pa., December 26–27, 1946.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., October, 1946
ILLINOIS, Peoria, May 9–10, 1947
INDIANA,
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Md., December 7, 1946
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May 3, 1947
NORTHERN CALIFORNIA, San Francisco, January 25, 1947
OHIO, Columbus, April 3, 1947
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Philadelphia, November 30, 1946
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Claremont, March 8, 1947
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Madison, May, 1947

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